Key takeaway

• **Inverse function Theorem**: Let $f(x)$ be a function that is both invertible and differentiable. Let $y=f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'(f^{-1}(x))\neq 0$,

$$\frac{dy}{dx}=\frac{d}{dx}\left(f^{-1}(x)\right)=\frac{1}{f'(f^{-1}(x))}$$

Alternatively, if $y=g(x)$. is the inverse of $f(x)$, then

$$g'(x)=\frac{1}{f'(g(x))}.$$

Derivatives of Inverse trigonometric Functions

① $\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}$

② $\frac{d}{dx}\cos^{-1}(x)=\frac{-1}{\sqrt{1-x^2}}$

③ $\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2}$

④ $\frac{d}{dx}\cot^{-1}(x)=\frac{-1}{1+x^2}$

⑤ $\frac{d}{dx}\sec^{-1}(x)=\frac{1}{|x|\sqrt{x^2-1}}$

⑦ $\frac{d}{dx}\csc^{-1}(x)=\frac{-1}{|x|\sqrt{x^2-1}}$

Derivative of the Exponential Function

• Derivative of the natural exponential function: Let $E(x)=e^x$ be the natural exponential function. Then, $E'(x)=e^x$.

In general, $\frac{d}{dx}(e^{g(x)})=g'(x)e^{g(x)}$.

• Derivative of the natural logarithmic function: If $x>0$ and $y=\ln(x)$, then

$$\frac{dy}{dx}=\frac{1}{x}$$

More generally, let $g(x)$ be the differentiable function. For all values of x for which $g'(x)>0$, the derivative of $h(x)=\ln(g(x))$ is given by:

$$h'(x)=\frac{1}{g(x)}g'(x).$$

• The derivative of General Exponential and logarithm function ① Let $b>0, b\neq 1$, and let $g(x)$ be a differentiable function. If $y=\log_b x \Rightarrow$

$$\frac{dy}{dx}=\frac{1}{x\ln(b)}.$$

Logarithmic Differentiation (Problem. Solving Strategies)

① To differentiate $y=h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln(y)=\ln(h(x))$.

② Use the properties of logarithms to expand $\ln(h(x))$ as much as possible.

③ Differentiate both sides of the equation. On the left, we will have $\frac{1}{y}\frac{dy}{dx}$.

④ Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.

⑤ Replace y by $h(x)$.

Example. we will derive this:

Let $y=b^x$. Find $\frac{dy}{dx}$.

① $y=b^x$

$\Rightarrow \ln(y)=\ln(b^x)$

② $\ln(y)=x\ln(b)$.

③ Differentiate with respect to x.

$$\frac{1}{y}\frac{dy}{dx}=\ln(b)$$

④ $y\cdot\left(\frac{1}{y}\frac{dy}{dx}\right)=y\cdot\ln(b)$

$\Rightarrow \frac{dy}{dx}=y\ln(b)$

⑤ Since $y=b^x$,

$$\frac{dy}{dx}=b^x\ln(b).\ \#$$

Implicit Differentiation

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

① Take the derivative of both sides of the equation. Keep in mind that y is a function of x.

② Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$, $h'(x) = \frac{g'(x)}{g(x)\ln(b)}$

(2) If $y = b^x \Rightarrow \frac{dy}{dx} = b^x \ln(b)$. More generally, if $h(x) = b^{g(x)}$, then $h'(x) = b^{g(x)} g'(x) \ln(b)$.

dx

(3) Factor out $\frac{dy}{dx}$ on the left.

(4) Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Name:

Recitation Instructor:

Recitation Time:

Homework #6 is due at 5:00 PM on Feb. 27 in your *recitation's* homework box near Cardwell 120.

1. Calculate the following derivatives. You don't need to show your work, but by writing out your work, you might make fewer mistakes.

i.e. same as $\tan^{-1}(12x)$

A. $\frac{d}{dx}\left(e^{5x} - 2^x + \arctan(7x + 5x)\right)$ sum and difference rule.

$$= \frac{d}{dx}(e^{5x}) - \frac{d}{dx}(2^x) + \frac{d}{dx}\operatorname{arctan}(12x)$$

$$= 5e^{5x} - 2^x \ln(2) + \frac{1}{1+(12x)^2}(12)$$

B. $\frac{d}{dt}\left(e^t \cdot \sin^{-1}(t^3)\right)$ product Rule.

$$= \left(\frac{d}{dt}e^t\right)\sin^{-1}(t^3) + e^t \frac{d}{dt}\left[\sin^{-1}(t^3)\right]$$

$$= e^t \sin^{-1}(t^3) + e^t\left(\frac{1}{\sqrt{1-(t^3)^2}}\right)(3t^2)$$

C. $\frac{d}{dw}\left(\frac{\ln(w^2) + \log_2(w)}{\tan(w)}\right) = \frac{\left[\frac{d}{dw}\ln(w^2)\right]\log_2(w) - \ln(w^2)\left[\frac{d}{dw}\log_2(w)\right]}{(\tan(w))^2}$

Quotient Rule.

$$= \frac{\left[\left(\frac{1}{w^2}\right)(2w)\right]\log_2(w) - \ln(w^2)\frac{1}{w\ln(2)}}{\tan^2(w)}$$

D. $\frac{d}{dx}\ln(\underbrace{\ln(\ln(x))}_{f(x)}) = \frac{1}{\ln(\ln(x))} \cdot \frac{d}{dx}\left[\ln(\underbrace{\ln(x)}_{g(x)})\right]$

$$\frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)} \cdot g'(x)$$

$$= \frac{1}{\ln(\ln(x))}\left[\frac{1}{\ln(x)} \cdot \frac{d}{dx}\ln(x)\right]$$

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} f'(x)$$

$$= \frac{1}{\ln(\ln(x))}\left(\frac{1}{\ln(x)}\right)\left(\frac{1}{x}\right).$$

2. Suppose that the height of a plane t minutes after take off is $h(t) = 2500\ln(t-1)$ feet for $0 \le t \le 15$. How fast is the plane rising 5 minutes after take off?

↳ This means we want to calculate the $\frac{dh}{dt}$ at $t=5$

$h(t) = 2500\ln(t-1)$

$$\frac{dh(t)}{dt} = \frac{d}{dt}\left[2500\ln(t-1)\right] = 2500\left[\frac{d}{dt}\ln(t-1)\right] = 2500\left(\frac{1}{t-1}\right) = \frac{2500}{t-1}.$$

3. The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1964 is modeled by the function

$$N(t) = 5e^{0.93t^2 - 0.7t} \qquad (0 \le t \le 4)$$

where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

A. Find and interpret the meanings of $N(0)$ and $N(4)$.

$N(0) = 5e^{0.93(0)^2 - 0.7(0)}$

$= 5e^0 = 5.$

∴ $N(0)$ is the number of cases (in thousands) with $t=0$ corresponding to beginning of 1960.

∴ $N(0)$ ~~is the sum at~~ tells us at 1960, there were 5000 cases.

Same interpretation for $N(4)$.

B. Find and interpret the meanings of $N'(0)$ and $N'(4)$.

Hint: The derivative calculates the rate of change of the function.

So, $N'(t)$ tells us the rate of change of the cases at time t.

Thus, $N'(0)$ and $N'(4)$ tells us if the number of cases is increasing or decreasing.

Find the specific values

4. Use logarithmic differentiation to find the following derivatives.

A. $\frac{d}{dx}x^{\cancel{2}3\cos(x)}$

Let $y = x^{3\cos(x)}$.

$\Rightarrow \ln(y) = \ln(x^{3\cos(x)})$

$\Rightarrow \ln(y) = 3\cos(x)\ln(x)$

Differentiating with respect to x, we get

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[3\cos(x)\ln(x)\right]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -3\sin(x)\ln(x) + \frac{3\cos(x)}{x}$

$\Rightarrow \frac{dy}{dx} = -3y\sin(x)\ln(x) + \frac{3y\cos(x)}{x}$

Since $y = x^{3\cos(x)}$,

$\frac{dy}{dx} = -3x^{3\cos(x)}\sin(x)\ln(x) + \frac{3x^{\cos(x)}\cos(x)}{x}$.

B. $f'(x)$ if $f(x) = \dfrac{e^{3x^2}(92x+3)^7}{(9x^6+5x^2)^3}$

$\to$ Use quotient rule.

$h(x) = \frac{f(x)}{g(x)} \Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

$\to$ To differentiate e^{3x^2}, use:

$y = e^{g(x)} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$.

5. A. Find $y' = \dfrac{dy}{dx}$ for the curve $x^2y^3 = (x^{\cancel{2}3} + y^{\cancel{2}3} - 1)^3$.

Use implicit differentiation.

Differentiating with respect to x,

$\frac{d}{dx}(x^2y^3) = \frac{d}{dx}\left[(x^3+y^3-1)^3\right]$

$2xy^3 + x^2\left(3y^2\frac{dy}{dx}\right) = 3(x^3+y^3-1)^{3-1}\left[\frac{d}{dx}(x^3+y^3-1)\right]$

$\Rightarrow 2xy^3 + 3x^2y^2\frac{dy}{dx} = 3(x^3+y^3-1)^2\left[3x^2+3y^2\frac{dy}{dx}\right]$

$\Rightarrow \frac{2xy^3}{3(x^3+y^3-1)^2} - 3x^2 = 3y^2\frac{dy}{dx} - \frac{3x^2y^2}{3(x^3+y^3-1)^2}\frac{dy}{dx}$

$\Rightarrow \frac{2xy^3}{3(x^3+y^3-1)^2} - 3x^2 = \frac{dy}{dx}\left[3y^2 - \frac{3x^2y^2}{3(x^3+y^3-1)^2}\right]$

$\therefore \frac{dy}{dx} = \left[\frac{2xy^3}{3(x^3+y^3-1)^2} - 3x^2\right] \Big/ \left[3y^2 - \frac{3x^2y^2}{3(x^3+y^3-1)^2}\right]$

B. Find the equation of the tangent line to $x^2y^3 = (x^2 + y^2 - 1)^3$ at $(1, 1)$.

$\hookrightarrow$ We did similar questions before.

6. Find $y' = \dfrac{dy}{dx}$ for the following curves.

A. $\tan(y) = \dfrac{x^2}{10}$

Use implicit Differentiation

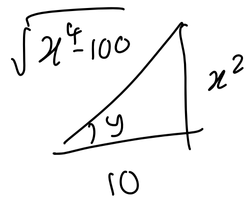
Let $\tan(y) = \dfrac{x^2}{10}$.

$\Rightarrow 10\tan(y) = x^2$

Differentiating with respect to x,

$10\sec^2(y)\dfrac{dy}{dx} = 2x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{10\sec^2(y)}$



Since $\tan(y) = \dfrac{x^2}{10}$

$\sec^2(y) = \dfrac{1}{\cos^2(y)} = \dfrac{1}{\cos(y)^2} = \dfrac{1}{\left(\frac{10}{\sqrt{x^4-100}}\right)^2}$

So, $\dfrac{dy}{dx} = \dfrac{2x}{10\left(\dfrac{1}{\left(\frac{10}{\sqrt{x^4-100}}\right)^2}\right)}$

B. $x^3 + xy + y^3 = 10$

C. $x^2y - e^y = x + 1$

D. $\cos(xy) = x^2$