Applications of derivative.

Motion along a Line

- Another use of the derivative is to analyze motion along the line.
- Let $s(t)$ be a function giving the position of an object at time t.

Velocity : $v(t) = s'(t) = \frac{ds}{dt}$

Speed : $|v(t)|$.

Acceleration of the object : $a(t) = v'(t) = s''(t)$.

Change in Cost and Revenue

→ If $C(x)$ is the cost of producing x items, then the marginal cost $MC(x)$ is $MC(x) = C'(x)$.

→ If $R(x)$ is the revenue obtained by selling x items, then the marginal profit $MP(x)$ is defined to be

$$MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x).$$

→ One application of derivatives is to estimate an unknown value of a function at a point by using a known value of a function at another given point together with its rate of change at the given point.

→ Recall that the derivative of $f(x)$ at $x = a$ is given by:

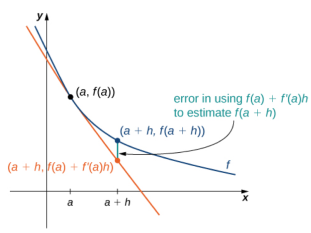
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

→ For small enough values of h,

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

∴ we can estimate $f(a+h)$ by

$$f(a+h) \approx hf'(a) + f(a).$$



Derivatives of the Sine and Cosine Function

→ $\frac{d}{dx}(\sin(x)) = \cos(x)$

→ $\frac{d}{dx}(\cos(x)) = -\sin(x)$

→ $\frac{d}{dx}(\tan(x)) = \sec^2(x)$

→ $\frac{d}{dx}\cot(x) = -\csc^2(x)$

→ $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

→ $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$

Chain Rule

→ When you want to differentiate a composition of function.

Intuition: Look at the example $h(x) = \sin(x^3)$.

- We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x.
- Consequently, we want to know how $\sin(x^3)$ changes as x changes.
- We can think of this as a chain reaction:

As x changes, x^3 changes leads to a change in $\sin(x^3)$.

- This chain reaction gives us hints as to what is involved in computing the derivatives of $\sin(x^3)$.

⇒ First of all, a Δ in x forces a Δ in x^3, so the derivative of x^3 is involved.

⇒ A Δ in x^3 leads to a change in $\sin(x^3)$. ∴ the derivative of $\sin(u)$ with respect to u, where $u = x^3$, is also part of the final derivative.

→ More formally,

So, $\sin(x^3) = h(x) = (f \circ g)(x)$ where $g(x) = x^3$, $f(x) = \sin(x)$.

∴ $h'(x) = f'(g(x))\,g'(x)$.

which is precisely the chain rule.

$$h'(a) = \lim_{x\to a} \frac{\sin(x^3)-\sin(a^3)}{x-a}$$

$$= \lim_{x\to a}\left(\frac{\sin(x^3)-\sin(a^3)}{x^3-a^3}\right)\left[\frac{x^3-a^3}{x-a}\right]$$

$$= \underbrace{\lim_{x\to a}\left(\frac{\sin(x^3)-\sin(a^3)}{x^3-a^3}\right)}\lim_{x\to a}\left(\frac{x^3-a^3}{x-a}\right)$$

Set $u=x^3$. as $x\to a$, $u\to a^3$.

$$\therefore = \lim_{u\to a^3}\frac{\sin(u)-\sin(a^3)}{u-a^3}\lim_{x\to a}\left(\frac{x^3-a^3}{x-a}\right)$$

$$= \left(\frac{d}{du}(\sin(u))\Big|_{u=a^3}\right)\left(\frac{d}{dx}(x^3)\Big|_{x=a}\right)$$

Name:

Recitation Instructor:

Recitation Time:

Homework #5 is due at 5:00 PM on Feb. 20 in your *recitation's* homework box near Cardwell 120.

1. Suppose that you run a toy factory. The total cost of producing x toys is $C(x) = 50,00\ \ + 2x$ dollars, and the total revenue generated by selling x toys is $R(x) = 10x - \ \ 0.1x^2$ dollars.

A. Find $R'(x)$, and interpret its meaning.

$$R'(x) = \frac{d}{dx}R(x)$$
$$= \frac{d}{dx}(10x - 0.1x^2) = 10 - 0.2x$$

↳ Marginal Revenue. Rate of Δ of revenue at x.

B. Find $C'(x)$, and interpret its meaning.

$$C'(x) = \frac{d}{dx}C(x) = \frac{d}{dx}(5000 + 2x) = 2.$$

Marginal Cost, i.e. rate of Δ of cost at x.

C. Find $\underbrace{\frac{d}{dx}(R(x) - C(x))}$, and interpret its meaning.

Marginal profit. Rate of Δ of profit at x.

Apply standard differentiation techniques

OR use (a) n (b).

$$\frac{d}{dx}(R(x)-C(x))$$
$$= \frac{d}{dx}R(x) - \frac{d}{dx}C(x)$$
$$= R'(x) - C'(x)$$
$$= 10 - 0.2x - 2$$

D. When is the derivative you found in Part C positive, and when is it negative? $= 8 - 0.2x$

$$8 - 0.2x > 0$$
$$\iff 8 > 0.2x$$
$$\iff \frac{8}{0.2} > x.$$

$$8 - 0.2x < 0$$
$$\iff 8 < 0.2x$$
$$\iff \frac{8}{0.2} < x.$$

E. If your company wants to maximize total profits, how many toys should they produce? What is the maximum total profit? Explain your reasoning.

↳ Total Profit $P(x) = R(x) - C(x)$

$$= 10 - 0.1x^2 - 5000 - 2x$$

$$= \underbrace{-\cancel{4}990 - 2x - 0.1x^2}$$

Draw this graph and find the Peak.

2. Suppose that a waiter brings you a cup of hot tea. Let $F(t)$ denote the temperature in degrees Fahrenheit of the tea after t minutes. Is $F'(3)$ positive or negative? Explain your answer.

→ $F'(3)$ is the rate of Δ of the temperature of the hot tea at time $t=3$.

Since the tea is cooling down, temperature is decreasing. ∴ the rate of Δ of temperature is falling. ∴ $F'(3)<0$.

3. Let $W(t)$ denote the number of km^3 of water in Tuttle Creek Lake t years after January 1, 2000. What does it mean if $W'(18) = -.01$ km^3/year?

↳ Same logic as 2.

4. Find the equation of the tangent line to $y = 5\cos(x)$ at $x = \frac{\pi}{4}$.

↳ Did this in written homework 3.

5. Let $v(x) = x\cos(x)$.

A. Find $v'(x)$.

using product rule

↳ $v'(x) = \frac{d}{dx} x\cos(x) = \left(\frac{d}{dx}x\right)\cos x + x\left(\frac{d}{dx}\cos(x)\right)$

$= \cos(x) + x(-\sin(x))$

$= \cos(x) - x\sin(x)$.

B. Find $v''(x)$. (This means the derivative of $v'(x)$.)

$v''(x) = \frac{d}{dx} v'(x) = \frac{d}{dx}\left[\cos(x) - x\sin(x)\right] = \frac{d}{dx}\cos(x) - \frac{d}{dx}(x\sin(x))$

$= -\sin(x) - \left[\sin(x) + x(-\cos(x))\right]$

$= -\sin(x) - \sin(x) + x\cos(x)$

$= -2\sin(x) + x\cos(x)$

C. Find $v^{(3)}(x)$. (This means the derivative of $v''(x)$.)

↳ Differentiate $v''(x)$.

So, find $\frac{d}{dx} v''(x) = \frac{d}{dx}(-2\sin(x) + x\cos(x))$.

6. Calculate the following derivatives. You don't need to show work, but by writing out your work, you might make fewer mistakes.

A. $\dfrac{d}{dw}\left(\dfrac{9w^{5/7}+5w+4}{\sin(w)+14w^9}\right)$ $= \dfrac{\frac{d}{dx}\left[9w^{5/7}+5w+4\right](\sin(w)+14w^9) - (9w^{5/7}+5w+4)\frac{d}{dx}\left[(\sin(w)+14w^9)\right]}{\left[\sin(w)+14w^9\right]^2}.$

↳ using quotient rule.

B. $\dfrac{d}{d\theta}\left(\dfrac{\sin(\theta)\cdot\cos(\theta)}{\sqrt{\theta}}\right)$ → Same as (B).

C. $\dfrac{d}{dx}\left(4\cos^7(x)+97\sec(x)-78\tan^2(x)\right)$

↳ Use trigo formula.

D. $\dfrac{d}{dt}\left(\sqrt{t}\cdot\sec(t)\right)$ → Product Rule + trigo formula.

E. $\dfrac{d}{dx}\left(\sin(x)\cdot\cos(x)\cdot\tan(x)\right)$

↳ Use product Rule.

F. $\dfrac{d}{d\theta}\sin(\sin(\cos(7\theta)))$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	4	1	2
2	5	7	2	6
3	12	9	8	-1
4	21	8	10	-3

7. Using the table above, calculate $c'(1)$ if $c(x) = f(g(x))$.

$c(x) = f(g(x))$.

So, $c'(x) = f'(g(x))\, g'(x)$

$\Rightarrow c'(1) = f'(g(1))\, g'(1) = f'(4)\,(2) = 10(2) = 20.$

8. Using the Quotient Rule, show that $\frac{d}{d\theta}\cot(\theta) = -\csc^2(\theta)$.

$$\frac{d}{d\theta}\cot(\theta) = \frac{d}{d\theta}\frac{1}{\tan(\theta)} = \frac{\left(\frac{d}{d\theta}1\right)\tan\theta - 1\left(\frac{d}{d\theta}\tan\theta\right)}{\tan^2(\theta)}$$

$$= \frac{-\sec^2(\theta)}{\tan^2\theta} = -\left(\frac{1}{\cos^2\theta}\right)\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= -\frac{1}{\sin^2\theta}$$

$$= -\csc^2\theta.$$

9. Suppose that the level of carbon monoxide in a city's air may be modeled by the formula $\text{CO}(p) = \sqrt{.5p + 15}$ parts per million when the population is p thousand people. It is estimated that t years today, the city's population will be $p(t) = 100 + .04t^2$ thousand people. At what rate will the carbon monoxide level be changing with respect to time 5 years from now?

$CO(p) = \sqrt{\frac{1}{2}p + 15}.$

$p(t) = 100 + 0.04(t^2)$

Δ in time t $\xrightarrow[\text{to change in population}]{\text{leads}}$ Δ in $p(t)$ $\xrightarrow[\text{to change in carbon monoxide}]{\text{which leads}}$ Δ in $CO(p)$

$\therefore$ First method using chain rule

$$\frac{d\,CO(p)}{dt} = \frac{d\,CO(p)}{dp} \cdot \frac{dp}{dt}$$

$$= \left[\frac{1}{2}\left(\frac{p}{2}+15\right)^{-1/2}\left(\frac{1}{2}\right)\right]\left[0.08t\right]$$

$$= \frac{1}{4}\left(\frac{p}{2}+15\right)^{-1/2} 0.08(t)$$

2nd method.

$$t \longmapsto p(t) \longmapsto COCp)$$

$$\underbrace{COCp(t) := \sqrt{\frac{1}{2}(100+0.04t^2)+15}}$$

Via composition of functions.

Then, calculate $\dfrac{d\, COCp(t)}{dt}$ directly.