Some Announcements

- Exam 1 (Thursday)
  ↳ 2nd Feb.
  7.05pm - 8.20pm.
  ⇒ Seaton 0057.
  ⇒ No notes, books, calculator.

Name:

Recitation Instructor:

Recitation Time:

Homework #2 is due at 5:00 PM on Jan. 30 in your *recitation's* homework box near Cardwell 120.

1. Suppose that $\lim_{x\to3} f(x) = 3$ and $\lim_{x\to3} g(x) = 4$. Find the following quantities.

A. $\lim_{x\to3} (f(x) + 2g(x)) \overset{(3)}{=} \left(\lim_{x\to3} f(x)\right) + \left(\lim_{x\to3} 2g(x)\right)$

$\overset{(4)}{=} \lim_{x\to3} f(x) + 2\lim_{x\to3} g(x) = 3+2(4) = 11.$

B. $\lim_{x\to3} \dfrac{f(x)\cdot g(x)}{2+x} \overset{(6)}{=} \dfrac{\lim_{x\to3} f(x)\cdot g(x)}{\lim_{x\to3} (2+x)} \overset{(5)}{\underset{(3)}{=}} \dfrac{\left(\lim_{x\to3} f(x)\right)\left(\lim_{x\to3} g(x)\right)}{\left(\lim_{x\to3} 2\right) + \lim_{x\to3} (x)}$

$\overset{(2),(1)}{=} \dfrac{3\cdot 4}{2+3} = \dfrac{12}{5}.$

C. $\lim_{x\to3} (x^2+1)\sqrt{f(x)+g(x)} = \left(\lim_{x\to3} x^2+1\right)\left(\lim_{x\to3} \sqrt{f(x)+g(x)}\right)$

$= \left[\lim_{x\to3} x^2 + \lim_{x\to3} 1\right]\left[\lim_{x\to3} f(x)+g(x)\right]^{1/2}$

$= \left[\left(\lim_{x\to3} x\right)^2 + \lim_{x\to3} 1\right]\left[\left(\lim_{x\to3} f(x)\right)+\left(\lim_{x\to3} g(x)\right)\right]^{1/2}$

$= (3^2+1)(3+4)^{1/2}.$

2. Suppose that an object is at position $s(t) = t^2$ feet at time t seconds.

A. Find the **average velocity** of the object over a time interval from time 2 seconds to time t seconds.

Formula for average velocity $= \dfrac{\text{Total distance travelled}}{\text{total time taken}} = \dfrac{\text{Final dist - Initial dist}}{\text{End time - Initial time}}$

$= \dfrac{s(t)-s(2)}{t-2}$

$= \dfrac{t^2-4}{t-2} = \dfrac{(t+2)(t-2)}{t-2} = t+2.$

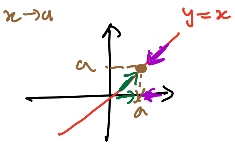
B. Find the **instantaneous velocity** of the object at time 2 seconds by taking the limit of the average velocity in Part A as $t \to 2$.

As $t\to2$, $t+2\to4$.

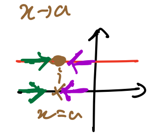
∴ Instanteous velocity is 4.

Basic limit Results

(1) $\lim_{x\to a} x = a.$



(2) $\lim_{x\to a} c = c.$



(3) $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(4) $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$

(5) $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

(6) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

(7) $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$

(8) Theorem: If $f(x)$ is a polynomial, then $\lim_{x\to a} f(x) = f(a).$

(9) If $f(x), g(x)$ are polynomials with $g(a) \neq 0 \Rightarrow \lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}.$

Similar to Qn2.

3. Suppose that an object is at position $s(t) = \dfrac{1}{t}$ feet at time t seconds.

A. Find the **average velocity** of the object over a time interval from time 1 second to time t seconds.

B. Find the **instantaneous velocity** of the object at time 1 second by taking the limit of the average velocity in Part A as $t \to 1$.

Similar to Qn 2.

4. Suppose that an object is at position $s(t) = \sqrt{t}$ feet at time t seconds.

A. Find the **average velocity** of the object over a time interval from time 4 seconds to time $4 + h$ seconds.

B. Find the **instantaneous velocity** of the object at time 4 seconds by taking the limit of the average velocity in Part A as $h \to 0$. (Note, $4 + h \to 4$ as $h \to 0$.)

5. Calculate the following limits using continuity.

A. $\displaystyle\lim_{x\to 3}\frac{6x}{x-1}$

• Note that the function $\frac{6x}{x-1}$ is defined everywhere in $\mathbb{R}$ except $x=1$. $\therefore$ it is continuous at $x=3$.

$$\lim_{x\to 3}\frac{6x}{x-1} = \frac{6(3)}{3-1} = \frac{18}{2} = 9 \#.$$

B. $\displaystyle\lim_{\theta\to\pi}\frac{\cos(\theta)}{\theta^4+1}$

• Note that the function $\frac{\cos(\theta)}{\theta^4+1}$ is defined everywhere in $\mathbb{R}$ since $\theta^4+1>0, \forall\theta\in\mathbb{R}$.

$$\therefore \lim_{\theta\to\pi}\frac{\cos(\theta)}{\theta^4+1} = \frac{\cos(\pi)}{\pi^4+1}$$

C. $\displaystyle\lim_{t\to 1}\left(\arctan(t)\cdot(t+1)^4\right)$

• Note that $\arctan(t)$ is continuous everywhere in $\mathbb{R}$.
• Note that $(t+1)^4$ is continuous everywhere.

$$\therefore \lim_{t\to 1}\left(\arctan(t)(t+1)^4\right) = \arctan(1)(1+1)^4 = 2^4\arctan(1).$$

6. Define the function $q(x)$ by

$$q(x) = \begin{cases} \sqrt{\cos(x)+8}, & x>0 \\ 3, & x=0 \\ e^x, & x<0 \end{cases}.$$

Where is $q(x)$ continuous? Wherever $q(x)$ is discontinuous, is it continuous from the left or continuous from the right?

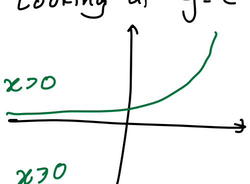
At $x>0$

Note that we can write $q(x)=\sqrt{\cos(x)+8}$, $x>0$ as $f(x)=\cos(x)+8$, $x>0$; $g(x)=\sqrt{x}$, $x>0$ $\Rightarrow (g\circ f)(x)=q(x)$ and the composition exist as Range(f) = domain (g).

Since $f(x)$ & $g(x)$ are ctn, and we know the composition of ctn functions are ctn, $q(x)$, $x>0$ is ctn!!

At $x<0$

Looking at $y=e^x$



intuitively we can see that e^x is continuous on its domain.

$\therefore \forall x<0$, e^x is continuous.

At $x=0$

To see if $q(x)$ is continuous, at $x=0$, we need to check the following:

① $f(0)$ is defined. → Obvious. From definition of function given, $q(0)=3$.

② The $\lim_{x\to 0} f(x)$ exist. → ie $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$.

③ $f(0)=\lim_{x\to 0} f(x)$.

$$\underbrace{\lim_{x\to 0^+}}_{\text{0 from right hand side}} q(x) = \lim_{x\to 0^+}\sqrt{\cos(x)+8} = \left(\lim_{x\to 0^+}\cos(x) + \lim_{x\to 0^+} 8\right)^{1/2}$$

$$= (1+8)^{1/2} = \sqrt{9} = 3 \text{ or } -3.$$

(rej -3 as $-1\le\cos(x)\le 1$ and then $7\le\cos(x)+8\le 9$. $\Rightarrow \sqrt{7}\le\cos(x)+8\le 3$.)

$$\underbrace{\lim_{x\to 0^-}}_{\text{limit from Left hand side.}} q(x) = \lim_{x\to 0^-} e^x = e^0 = 1.$$

$$\therefore \lim_{x\to 0^+} q(x) \neq \lim_{x\to 0^-} q(x).$$

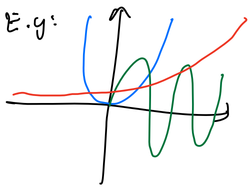
$\therefore$ ② fails.

$\therefore \exists$ a discontinuity at $x=0$.

Notes on Continuity and Discontinuity

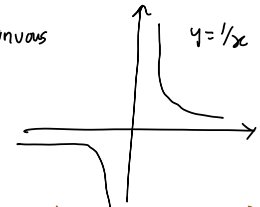
Intuition on continuity: Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such a function is called continuous.

E.g.:



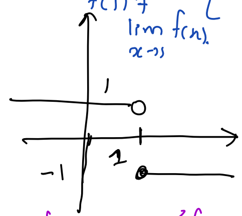
Example of non-continuous function:

called an infinite discontinuity



Here, at $x=0$, $f(x)$ is undefined.

Jump Discontinuity.



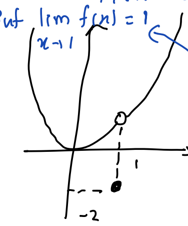
Here, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

So, $\lim_{x\to 1} f(x)$ does not exist.

So, $f(1) \neq \lim_{x\to 1} f(x)$

removable continuity.

In this case, $f(1) = -2$, But $\lim_{x\to 1} f(x) = 1$



$f(x) = \begin{cases} x^2, & \text{if } x \neq 1 \\ -2, & \text{if } x = 1 \end{cases}$

- All 3 of this functions have breaks in their graph.
- The point at which the break happens is said to have a discontinuity at a point.

- Based on how discontinuity happens above, we can see that for $f(x)$ to be continuous at $x=a$:

Problem-Solving Strategy: To determine if f is continuous at $x=a$, follow these 3 steps.

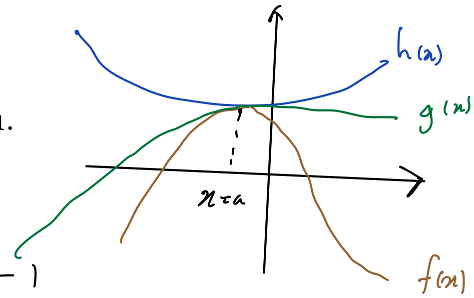
1. $f(a)$ needs to be defined.
2. The limit $\lim_{x\to a} f(x)$ must exist. ⇐ i.e. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.
3. $\lim_{x\to a} f(x) = f(a)$.

∴ $f(x)$ is continuous at $x=a$ if all 3 of the condition is satisfied above, and $f(x)$ is discontinuous at $x=a$ if any 1 of 3 condition fails.

Some Theorems

1. Polynomials and rational functions are continuous.
2. Trigonometric functions are continuous.
3. Composition of Continuous function is continuous.

The squeeze theorem: Let $f(x)$, $g(x)$ and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If $f(x) \leq g(x) \leq h(x)$ $\forall x \neq a$ in the open interval containing a and $\lim_{x\to a} f(x) = L = \lim_{x \to a} h(x) \Rightarrow \lim_{x\to a} g(x) = L$.

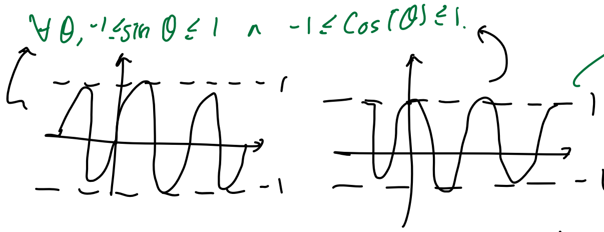


7. Calculate the following limits using the squeeze theorem.

A. $\lim_{\theta\to 0} \theta \cos\left(\frac{3}{\theta}\right)$

The trick with $\sin(\theta)$ / $\cos(\theta)$:

$\forall \theta, -1 \leq \sin\theta \leq 1 \;\wedge\; -1 \leq \cos(\theta) \leq 1$.



$\therefore \; -1 \leq \cos\left(\frac{3}{\theta}\right) \leq 1$

$\therefore -\theta \leq \theta\cos\left(\frac{3}{\theta}\right) \leq \theta$.

Note that $\lim_{\theta\to 0} -\theta = -\left(\lim_{\theta\to 0}\theta\right) = 0$

$\lim_{\theta \to 0} \theta = 0$.

$\therefore$ By squeeze theorem, $\lim_{\theta\to 0}\theta\cos\left(\frac{3}{\theta}\right) = 0$.

B. $\lim_{x\to 5} w(x)$ provided that the function $w(x)$ satisfies $10x - 25 \leq w(x) \leq x^2$ for all $x \neq 5$. (Sketching a graph might help you visualize this.)

Note that $\lim_{x\to 5}(10x-25) = 10\lim_{x\to 5} x - \lim_{x\to 5} 25 = 10(5) - 25 = 25$

$\lim_{x\to 5} x^2 = \left(\lim_{x\to 5} x\right)^2 = (5)^2 = 25$.

$\lim_{x\to 5} 10x - 25 = 25 = \lim_{x \to 5} x^2$.

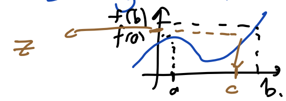
$\therefore$ By squeeze theorem, $\lim_{x\to 5} w(x) = 25$ #.

8. Find the following limits using special trigonometric limits combined with limit laws.

A. $\lim_{\theta\to 0} \frac{2\theta}{5\sin(\theta)}$

B. $\lim_{\theta\to 0} \frac{(\theta^2+7)\cdot(1-\cos(\theta))}{\theta}$

IVT: Let f be continuous over a closed, bounded interval $[a,b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ satisfying $f(c) = z$.



9. Using the Intermediate Value Theorem, show that $2\cos(x) - \sin(x)$ has a root in the interval $\left[0, \dfrac{\pi}{2}\right]$.

- Note that the IVT tells us that for any real number z between $f(a)$ and $f(b)$ in the y-axis, $\exists$ a real number c between a and b in the x-axis such that $f(c) = z$.
- Therefore, if we can show that one of the following values is positive and the other is negative (The values are $2\cos(0) - \sin(0)$, $2\cos(\pi/2) - \sin(\pi/2)$), then by IVT since 0 lies between these z values in the y-axis, there exist a corresponding value in the x-axis that maps to 0.

So, $2\cos(0) - \sin(0) = 1$ and $2\cos(\pi/2) - \sin(\pi/2) = 0 - 1 = -1$.

$\therefore$ 0 is between $[-1, 1]$.

$\therefore$ By IVT, $2\cos(x) - \sin(x)$ has a root in the interval $[0, \pi/2]$.

10. Let $f(x) = 5x^2$ and $g(x) = e^x$. Using the Intermediate Value Theorem, show that $f(x) = g(x)$ has a solution in the interval $[0, 1]$. (Note that $f(x) = g(x)$ exactly when $f(x) - g(x) = 0$.)

- Similar to Qn 9.

11. Suppose that the speedometer on your car reads 60 miles per hour at 1 PM and 70 miles per hour at 2 PM. Using math, argue why your speedometer must have read 65 miles per hour at some point between 1 PM and 2 PM.