Mahler Measure in The Fuglede-Kadison Determinant

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Abstract

In this paper, we will talk about how Mahler Measure arises in the study of Fuglede-Kadison determinant. The Fuglede-Kadison Determinant is the generalisation of determinants in the Finite Dimensional Hilbert Spaces to the infinite dimensional spaces. One such instance where this occurrence arises naturally is in the study of the multiplication operators in the Hilbert space of all complex-valued functions on the unit circle. In order to understand how the Mahler Measure arises in the study of Fuglede-Kadison determinants in this space, our methods involve the use of the Group Von Neumann Algebra, Von Neumann trace, spectral family and the spectral density function of the multiplication operator.

1 Introduction

In this introduction, we will show the flow of the paper. Firstly, we will define the tools needed to we will be discussing in this paper: $L^2(S^1)$ Hilbert Space, self-adjoint operators, the multiplication operator we will be studying in the $L^2(S^1)$ Hilbert Space specifically, Von Neumann Algebra, Von Neumann Trace, Spectral Family and Spectral Density Function and finally establishing the connections between the Mahler Measure and Fuglede-Kadison determinant.

2 Key Definitions

The $L^2(S^1)$ is the Hilbert Space of all complex-valued functions on the unit circle. It is a class of functions with which Fourier series are most naturally associated with. As an Hilbert Space, $L^2(S^1)$ is a complete inner product space with the associated inner product:

$$\langle x(t), y(t) \rangle = \int_0^{2\pi} \overline{x(t)} y(t) dt$$

$\forall x(t), y(t) \in L^2(S^1)$. The length of a function $x(t)$ is $\| x(t) \| = \int_0^{2\pi} \overline{x(t)} x(t) dt = \int_0^{2\pi} |x(t)|^2 dt$.

We shall now talk about self-adjoint hilbert operators in Hilbert Spaces.

Definition 2.1 *Self-Adjoint Hilbert Operators Let $T : H \to H$ be a bounded linear operator, where H is a hilbert space. Then, the hilbert-adjoint operator T^* of T is the operator:*

$$T^* : H \to H such that \langle Tx, y \rangle = \langle x, T^* y \rangle$$

$\forall x, y \in H$.

T is said to be self-adjoint if $T^* = T$. Also, note that $\| T^* \| = \| T \|$. Now, there is an important theorem we should know before proceeding:

Theorem 2.1 *Let $T : H \to H$ be a bounded linear operator on a Hilbert Space H. Then: (a) If T is self-adjoint, $\langle Tx, x \rangle$ is real $\forall x \in H$. (b) If H is complex and $\langle Tx, x \rangle$ is real $\forall x \in H$, the operator T is self-adjoint.*

Now, let's head on to the multiplicative operator M_g, where $g(x)$ is a bounded complex-valued function in $L^2(S^1)$. Firstly,

$$M_g : L^2(S^1) \to L^2(S^1).$$

However, we need to take note that M_g might not always be a self-adjoint operator. Therefore, given M_g, we can construct a self-adjoint operator $M_{g\widehat{g}}$. This will be shown in the following section.

2.1 Constructing Self-Adjoint Operator

Let us express M_g as $g(t) = \sum_{i=m}^{n} c_i t^i$ and M_{g^*} as $g^*(t)$.
We need to find M_{g^*} such that

$$\langle M_g(f_1), f_2 \rangle = \langle f_1, M_{g^*}(f_2) \rangle.$$

From the definition above, we see that, therefore the operator M_{g^*} has to satisfy the following expression:

$$\int_0^{2\pi} \overline{g(e^{2\pi i\theta}) f_1(\theta)} d\theta = \int_0^{2\pi} \overline{f_1(\theta) g(e^{2\pi i\theta}) f_2(\theta)}.$$

Therefore, $g^*(t) = \sum_{i=m}^{n} \overline{c_i} t^{-i}$. From here, we can obtain M_{gg^*} from $M_g M_{g^*}$. We shall denote gg^* as f for simplification. Thus, $M_{gg^*} = M_f$.

2.2 Group Von Neumann Algebra Von Neumann Trace

Definition 2.2 *Group von Neumann algebra The Group von Neumann algebra $\nu(G)$ is defined as the algebra of $G-equivariant$ bounded linear operators from $\ell^2(G)$ to $\ell^2(G)$*

$$\nu(G) := B(\ell^2(G))^G.$$

An important feature of the group von Neumann algebra is its standard trace.

Definition 2.3 *Von Neumann Trace The von Neumann trace on $\nu(G)$ is defined by*

$$tr_{\nu(G)} : \nu(G) \to \mathbb{C}, f \mapsto \langle f(e), e \rangle_{\ell(G)}$$

While the Group Von Neumann algebra and the Von Neumann trace are defined on the hilbert space $\ell^2(G)$, this is highly relevant to our case of $L^2(S^1)$ space due to the following fact. In our special case, the group G we will be dealing with is $\mathbb{Z}$. The Fourier transform yields an isometric $\mathbb{Z}-equivariant$ isomorphism $\ell^2(\mathbb{Z}) \cong \mathrm{L}^2(S^1)$. Hence,

$$\nu(\mathbb{Z}) = B(L^2(S^1))^{\mathbb{Z}}.$$

We obtain an isomorphism

$$L^\infty(S^1) \cong \nu(\mathbb{Z})$$

by sending $f \in L^\infty(S^1)$ to the $\mathbb{Z}-equivariant$ operator $M_f : L^2(S^1) \to L^2(S^1)$, $g \mapsto g \cdot f$, where $g \cdot f(x)$ is defined by $g(x)f(x)$. Under this identification, the trace becomes

$$tr_{\nu(\mathbb{Z})} : L^\infty(S^1) \to \mathbb{C}, f \mapsto \int_{S^1} f d\mu.$$

2.3 Spectral Family

Definition 2.4 *Spectral Family A real spectral family (or real decomposition of unity) is a one-parameter family $\xi = (E_\lambda)_{\lambda \in \mathbb{R}}$ of projections defined on the hilbert space H, which depends on a real parameter λ and is such that:*

$$E_\lambda \leq E_\mu (\lambda \leq \mu) \lim_{\lambda \to -\infty} E_\lambda x = 0 \lim_{\lambda \to \infty} E_\lambda x = x E_{\lambda+0} x = \lim_{\mu \to \lambda+0} E_\mu x = E_\lambda x (x \in H)$$

ξ is called the spectral family on an interval $[a, b]$ if $E_\lambda = 0$ for $\lambda < a$ and $E_\lambda = I$ for $\lambda > b$.

2.3.1 Spectral family for M_f

Now, we shall discuss the spectral density function for the operator M_f. It is the family of projections for the varying parameter λ:

$$E : \lambda \to M_{\chi_{\{\theta \in [0,2\pi] | f(\theta) < \lambda}}.$$

An example will be given in the images. (Add these to the appendix too)

3 Spectral Density function

Definition 3.1 *Spectral Density Function Let U and V be Hilbert $\nu(G)$-modules. Let $f : dom(f) \subset U \to V$ be a $G-$equivariant closed densely defined operator. Then, for $\lambda \in \mathbb{R}$ the spectral projection $E^{f^* f}_{lambda^2}$ is G-equivariant and*

$$F(f)(\lambda) = dim_{\nu(G)}(im(E^{f^* f}_{lambda^2})).$$

Thus, for our case in $L^2(S^1)$,

$$\begin{aligned} F(f)(\lambda) &= dim_{\nu(G)}(im(E^{f^* f}_{lambda^2})) \\ &= tr_{\nu(\mathbb{Z})}(E^{f^* f}_{lambda^2}) \\ &= \langle E^{f^* f}_{lambda^2}(1t^0), 1t^0) \rangle \\ &= \int \chi_{\{\theta \in [0,2\pi] | f(\theta) < \lambda} d\theta \end{aligned}$$

This is essentially the length of the interval where $f(e^{2\pi i \theta}) < \lambda^2$. An example will be given in the images. (Add these to the appendix too)

4 Fuglede-Kadison Determinant

Definition 4.1 *Fuglude-Kadison Determinant Let $f : U \to V$ be a morphism of finite dimensional Hilbert $\nu(G)$-modules with spectral density function $F = F(f)$. Define its (generalised) Fuglede-Kadison determinant*

$$det_{\nu(G)}(f) \in [0, \infty)$$

by $det_{\nu(G)}(f) := exp(\int_{0^+}^{\infty} \ln(\lambda) dF > -\infty$, otherwise 0.

We also know that the determinant of the multiplication matrix can be defined through the Mahler Measure, as follows:

$$det(M_f) = exp(\int_{S^1} \ln|f(z)| \cdot \chi_{\{u \in S^1 | f(u) \neq 0\}} dvol_Z)$$

where the expression on the right hand side is the mahler measure.

Thus, we can see that

$$\int_{0^+}^{\infty} \ln(\lambda) dF = \int_{S^1} \ln|f(z)| \cdot \chi_{\{u \in S^1 | f(u) \neq 0\}} dvol_Z$$