Knot Invariants via Algebraic Topology and Functional Analysis

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Abstract

In this paper, we will talk about how Mahler Measure arises in the study of Fuglede-Kadison determinant. The Fuglede-Kadison Determinant is the generalisation of determinants in the Finite Dimensional Hilbert Spaces to the infinite dimensional spaces. One such instance where this occurrence arises naturally is in the study of the multiplication operators in the Hilbert space of all complex-valued functions on the unit circle. In order to understand how the Mahler Measure arises in the study of Fuglede-Kadison determinants in this space, our methods involve the use of the Group Von Neumann Algebra, Von Neumann trace, spectral family and the spectral density function of the multiplication operator.

First, we will establish that the operator is a self-adjoint hilbert operator, otherwise we need to construct one such operator from the operator given. Next, we need to find the spectral family of the operator and use it to build the spectral density function. The spectral density function is used to define the Generalised Fuglede-Kadison determinant, and then we will see how the Mahler Measure arises in the determinant.

Keywords**—** Fuglede-Kadison Determinant, Mahler Measure

1 $\ell^2(\mathbb{Z})$ and the Right Multiplication Operator $R_{p(t)}$

The Hilbert Space $\ell^2(\mathbb{Z})$ is defined as:

$$\ell^2(\mathbb{Z}) = \{\sum_{n=1}^{\infty} C_n[n] | \sum_{i=1}^{\infty} |C_i|^2 < \infty\}$$

where $C_i \in \mathbb{C}$.
Note that elements in $\ell^2(\mathbb{Z})$ can be denoted as polynomials in this manner:

$$\sum_{n=1}^{\infty} C_n[n] \to \sum_{n=1}^{\infty} C_n t^n$$

where t acts as an empty parameter.
The linear operator we are interested in is the Right shift/transform operator $R_{p(t)}$, where $p(t) \in \ell^2(\mathbb{Z})$.

$$\begin{gathered} R_{p(t)} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \\ \sum_{h \in \mathbb{Z}} C_h[h] \mapsto \sum_{h \in \mathbb{Z}} C_h[hg] \end{gathered}$$

In terms of polynomials,

$$\sum_{h\in\mathbb{Z}} C_h t^h \mapsto \sum_{h\in\mathbb{Z}} C_h t^{hg}.$$

The determinant of the linear operator $R_{p(t)}$, $det(R_{p(t)})$ is the Fuglede-Kadison determinant. However, we can also prove that the $det(R_{p(t)})$ is the same as the Mahler measure of the polynomial, $p(t)$, which we will prove as we go on.

2 $\ell^2(\mathbb{Z}) \cong L^2(S^1)$

The $\ell^2(\mathbb{Z})$ Hilbert Space is isomorphic to the Hilbert space of all complex-valued functions on the unit circle, $L^2(S^1)$ [8] by fourier transform. We can define the transform in the following manner:

$$\sum_{n\in\mathbb{Z}} C_n t^n \mapsto (z \in \mathbb{C} \mapsto \sum_{-\infty}^{\infty} C_n z^n)$$

where the expression in the brackets on the right hand side denotes a function in $L^2(S^1)$.
As $\ell^2(\mathbb{Z}) \cong L^2(S^1)$, there is an analogous right shift linear operator, $M_g \in L^\infty$, in $L^2(S^1)$. M_g denotes the multiplication of every function $f \in L^2(S^1)$ with function $g \in L^2(S^1)$, with the standard function multiplication operation, $(f \cdot g)(x) = f(x)g(x)$. The set of all the bounded linear operators in $L^2(S^1)$ is denoted by L^∞. This transform will aid us to prove that the $R_{p(t)}$, $det(R_{p(t)})$, as transforming to an isomorphic space allows us to tap into a different set of tools which are easier to use.

3 Establishing the Self-adjoint Hilbert Operator

From here on forth, we will focus on $L^2(S^1)$. Let the multiplication of function $g \in L^2(S^1)$ bounded operator be denoted by M_g.

$$M_g : L^2(S^1) \to L^2(S^1).$$

We need to take note that M_g might not always be a self adjoint hilbert operator and Spectral Families can only be constructed for self-adjoint bounded linear operators [1]. Therefore, We need to figure out how to construct a self-adjoint hilbert operator if the given operator M_g does not satisfy the given condition above.
M_g is said to be a self-adjoint operator if $\forall f_1, f_2 \in L^2(S^1)$, $\langle (M_g)(f_1), f_2\rangle = \langle f_1, (M_g)(f_2)\rangle$ [2].
Suppose M_g is not self-adjoint.
Thus, we need to find M_{g^*} such that

$$\langle M_g(f_1), f_2\rangle = \langle f_1, M_{g^*}(f_2)\rangle.$$

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Let us express M_g as $g(t) = \sum_{i=m}^{n} c_i t^i$ and M_{g^*} as $g^*(t)$.
From the definition above, we see that, therefore the operator M_{g^*} has to satisfy the following expression:

$$\int_0^{2\pi} \overline{g(e^{2\pi i\theta}) f_1(\theta)} f_2(\theta) d\theta = \int_0^{2\pi} \overline{f_1(\theta)} g(e^{2\pi i\theta}) f_2(\theta) d\theta.$$

From here, we can deduce that in order for $\langle M_g(f_1), f_2\rangle = \langle f_1, M_{g^*}(f_2)\rangle$, $\overline{g(e^{2\pi i\theta})} = g(e^{2\pi i\theta})$.
Therefore, $g^*(t) = \sum_{i=m}^{n} \overline{c_i} t^{-i}$.
From here, we can obtain M_{gg^*} from $M_g M_{g^*}$. We shall denote gg^* as f for simplification. Thus, $M_{gg^*} = M_f$.
We will be using M_f to denote the self-adjoint operator from this point on wards in this paper.

4 Spectral Family of M_f

A real spectral family (or real decomposition of unity) is a one-parameter family $\xi = (E_\lambda)_{\lambda \in \mathbb{R}}$ of projections defined on the hilbert space $L^2(S^1)$, which depends on a real parameter λ and is such that:

$$\begin{aligned} & E_\lambda \leq E_\mu \\ & \lim_{\lambda \to -\infty} E_\lambda x = 0 \\ & \lim_{\lambda \to \infty} E_\lambda x = x \\ & E_{\lambda+0} x = \lim_{\mu \to \lambda+0} E_\mu x = E_\lambda x \end{aligned}$$

when $when \lambda \leq \mu$ and $x \in L^2(S^1)$ [4].
ξ is called the spectral family on an interval $[a, b]$ if $E_\lambda = 0$ for $\lambda < a$ and $E_\lambda = I$ for $\lambda > b$ [4].
In our specific case of M_g in $L^2(S^1)$, although the elements are complex-valued functions on the unit circle, the spectral family is the family of projections for the varying parameter λ:

$$E : \lambda \to M_{\chi_{\{\theta \in [0,2\pi] | f(\theta) < \lambda\}}}.$$

The $\chi_{\{\theta \in [0,2\pi] | f(\theta) < \lambda\}}$ denotes the characteristic function which denotes the value of 1 to the range of the function f which falls below the parameter λ and 0 to the range of the function which is above the parameter λ. This results in a discontinuous function where the range is $\{0, 1\}$. We, then, multiply this characteristic function with f, whereby the range of the intervals of function f which falls below $lambda$ is preserved, whilst the rest of the function becomes 0. This is the spectral family for that specific λ parameter and we denote it by E_λ. (Take a look at the Appendix for a few examples).
Although $L^2(S^1)$ is a complex-valued function on a unit circle, the parameter λ falls on the interval $[m, M] \subset \mathbb{R}$, where $m = inf_{||x||=1} \langle M_g(f), f \rangle$ and $M = sup_{||x||=1} \langle M_g(f), f \rangle$ [4].

5 Spectral Density Function of M_f

Let U and V be Hilbert $\nu(G)$-modules. Let $f : dom(f) \subset U \to V$ be a $G - equivariant$ closed densely defined operator. Then, for $\lambda \in \mathbb{R}$ the spectral projection $E^{f^* f}_{lambda^2}$ is G-equivariant and

$$F(f)(\lambda) = dim_{\nu(G)}(im(E^{f^* f}_{lambda^2})).$$

Thus, for our case in $L^2(S^1)$,

$$\begin{aligned} F(f)(\lambda) &= dim_{\nu(G)}(im(E^{f^* f}_{lambda^2})) \\ &= tr_{\nu(\mathbb{Z})}(E^{f^* f}_{lambda^2}) \\ &= \langle E^{f^* f}_{lambda^2}(1t^0), 1t^0) \rangle \\ &= \int \chi_{\{\theta \in [0,2\pi] | f(\theta) < \lambda} d\theta \end{aligned}$$

This is essentially the length of the interval where $f(e^{2\pi i \theta}) < \lambda^2$ [5], where the trace is the Von Neumann Trace [6].

6 Linking Fuglede-Kadison Determinant and Mahler Measure

Let us denote $F(f)(\lambda)$ as F. Then, the Fuglede-Kadison determinant is the

$$det_{\nu(G)}(f) := exp(\int_{0^+}^{\infty} \ln(\lambda) dF)$$

if the determinant is more than negative infinity. Otherwise, the determinant of the multiplication by function f operator is 0 [7].

We also know that the determinant of the multiplication matrix can be defined through the Mahler Measure, as follows:

$$det(M_f) = exp(\int_{S^1} \ln |f(z)| \cdot \chi_{\{u \in S^1 | f(u) \neq 0\}} dvol_Z)$$

where the expression on the right hand side is the mahler measure.

Thus, we can see that

$$\int_{0^+}^{\infty} \ln(\lambda) dF = \int_{S^1} \ln |f(z)| \cdot \chi_{\{u \in S^1 | f(u) \neq 0\}} dvol_Z$$

7 Algorithm

In this section, we will use some computational tools to approximate the Spectral Density Function of a function and verify that the Mahler measure is indeed to the Fuglede-Kadison determinant. The results are added here while the code is added to the Appendix.

Here, we will need to use alternate definitions of Mahler Measure to compute it: Given $P \in \mathbb{C}[x]$, such that

$$P(x) = a \prod_i (x - \alpha_i)$$

define the Mahler Measure of P as

$$M(P) = |a| \prod_i max\{1, |\alpha_i|\}$$

[3].

References

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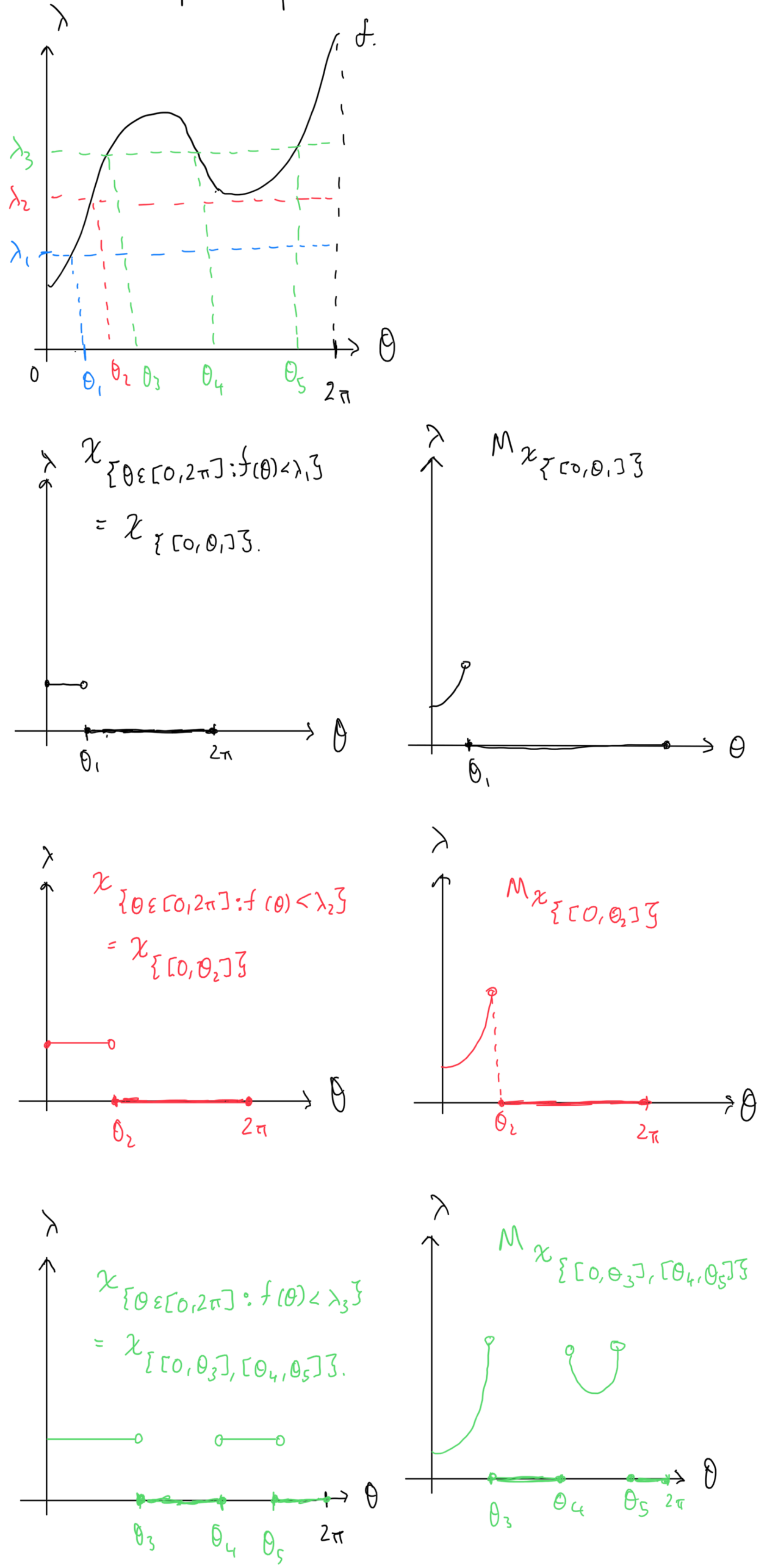
A Appendices

A.1 Characteristic Function and Spectral Family of a given function

Let f be a multiplication operator on $L^2(S^1)$.



A.2 Matlab Code to generate Spectral Density Function and Verify Mahler Measure and Determinant