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NANYANG TECHNOLOGICAL UNIVERSITY

DIVISION OF MATHEMATICAL SCIENCES
SCHOOL OF PHYSICAL & MATHEMATICAL SCIENCES

FINAL THESIS

Title: A Conjecture for the Eigenvalues of pseudo-Anosov Mappings of Surfaces

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of the requirements for the *Degree of Bachelor of Science*

Summary

1 Acknowledgements

Firstly, I would like to thank my supervisor, Assoc Prof Andrew Kricker, who has guided me through 2 Undergraduate Research Experience on Campus (URECA) projects, which has led up to this Final Year Project. Without him, none of this would have been possible.

Lastly, I would like to thank my family and friends, who have supported me in uncountably infinite ways throughout this journey.

2 Abstract

Let $\Sigma_{g,n}$ be an orientable surface of genus g and n punctures, such that $2-2g-n<0$. Let $f \in \text{Mod}(\Sigma_{g,n})$ denote an element in the mapping class group. The study of the bounds of $D(f)$ and $D_h(f)$ is a dynamic field in geometric topology.

Given any automorphisms f of a connected orientable surface $\Sigma_{g,n}$, it is evident that any virtual homological spectral radius for f is greater than or equal to 1. It is shown by C.T. McMullen that any virtual homological spectral radius for a pseudo-Anosov surface automorphism f is strictly lesser than the dilation if the invariant foliations for f have prong singularities of odd order [McM13]. Since the topology of the mapping torus depends only on the mapping class f, we denotes its topological type by N_f. A celebrated theorem by Thurston [Thu98] asserts that N_f admits a hyperbolic structure iff f is pseudo-Anosov. Kojiwa-Macshane proved that:

$$ln(D(f)) \geq \frac{Vol(M_f)}{3\pi|\chi(X)|},$$

where M_f, $\chi(X)$ and $D(f)$ are the mapping torus, Euler characteristic of the surface and the dilation of f respectively [Mcs15].

With references to these, Dr Thang Le conjectured relations between the spectral radius of $\widetilde{f}$ acting on $H_1(\widetilde{\Sigma_{g,n}}, \mathbb{Z})$, and the volume of the mapping torus with respect to f,

$$M_f := \frac{\mathbb{I} \text{ x } X}{(1,x)\sim(0,f(x))},$$

where $\widetilde{\Sigma_{g,n}}$ varies over the finite covers of $\Sigma_{g,n}$ to which f lifts.

In this thesis, I will first explain the construction of the finite type covers of $\Sigma_{g,n}$ via monodromy representation. The existence and construction of lift of f, $\widetilde{f}$, will be done by the standard theorems in Algebraic Topology, Combinatorial Group Theory and Fox Derivative Calculus respectively. We will obtain the spectral radius from each finite type cover from the collection of all covers constructed and test the conjectures. All these theory will be developed in this thesis, together with a brief introduction to Mapping Class Groups and Pseudo-Anosov Maps. Then, I will explain how this theory is translated to work on the code.

Results of the 4-sheeted cover will be explained thoroughly step-by-step and the results for 5 and 6 sheeted covers will also be included. Finally, I will explain the checks we have put in place to ensure the correctness of the algorithm before I conclude this thesis with suggestions for improvement.

3 Introduction

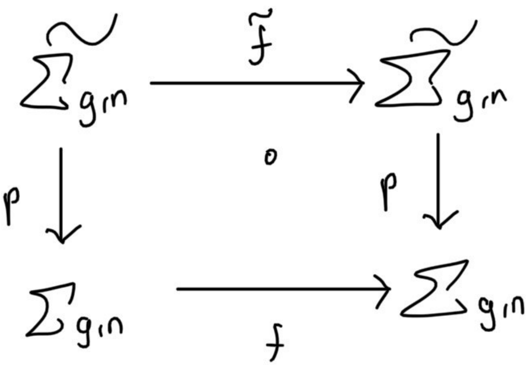
3.1 Thang Le's Conjectures

Let $\Sigma_{g,n}$ be an orientable surface of genus g with n punctures, such that:

$$\chi(\Sigma_{g,n}) = 2 - 2g - n < 0.$$

We will assume that $\Sigma_{g,n}$ has at least one boundary component, and that the genus and the number of punctures is finite. The Mapping Class Group of $\Sigma_{g,n}$, denoted as $\mathrm{Mod}(\Sigma_{g,n})$, is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,n}$.

For any Mapping Class Group of $\Sigma_{g,n}$ $f \in \mathrm{Mod}(\Sigma_{g,n})$, the homological spectral radius is defined to be the spectral radius of the induced linear automorphism of f on the first homology of Σ with complex coefficients. That is to say, it is the largest modulus for all complex eigenvalues of $f_* : H_1(\Sigma_{g,n}, \mathbb{C}) \to H_1(\Sigma_{g,n}, \mathbb{C})$. For any connected finite cover $\widetilde{\Sigma}$ of Σ, let the automorphism $\widetilde{f} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ be the lift of f.



In this case, the homological spectral radius for $\widetilde{f}$ is said to be the virtual homological spectral radius of f.

Let $D_h(f)$ be the supremum of all virtual homological spectral radius, based on the collection of all finite covers of $\Sigma_{g,n}$ and the mapping torus associated to a mapping class f is defined as $M_f = \frac{\Sigma \text{ x } \mathbb{I}}{(x,0)\sim(f(x),1)}$.

The purpose of this project is to develop an understanding of the conjecture made by Dr Thang Le and develop computational tools to test the conjecture. The conjecture made by Dr Thang Le suggests a a lower bound for logarithm of $D_h(f)$ using the hyperbolic volume of mapping torus M_f with respect to any pseudo-Anosov function f:

$$log(D_h(f)) \geq \frac{Vol(M_f)}{3\pi|\chi|}.$$

3.2 Framework and Motivation of this thesis

In this thesis, I will explain and develop the theoretical foundation that my software uses to verify Thang Le's bounds. For the purposes of my project, the surface we will specifically focus on is the once-punctured torus, $\Sigma_{1,1}$, where the number of genus $g = 1$ and the number of punctures $n = 1$, and the pseudo-Anosov map that we will use is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

The mapping torus of the pseudo-Anosov is well studied: It happens to be the figure-8 knot complement and it is well known that the hyperbolic volume of the figure-8 knot complement is 2.02988.

As the once-punctured torus, $\Sigma_{1,1}$ is an orientable surface of genus 1 with 1 puncture, the euler characteristic of the surface, $\chi(\Sigma_{1,1})$ is as follows:

$$\begin{aligned} \chi(\Sigma_{1,1}) &= 2 - 2(1) - (1) \\ &= -1. \end{aligned}$$

Then, modifying Thang Le's general conjecture to our class of space, he proposes that:

$$\begin{aligned} log(D_h(f)) &\geq \frac{M_f}{3\pi|\chi|} \\ &= \frac{2.02988}{3\pi|1|} \\ &= \frac{2.02988}{3\pi} \\ &= 0.21537695725. \end{aligned}$$

Thus, we are required to show the following:

$$\begin{aligned} D_h(f) &\geq e^{0.21537695725} \\ &= 1.24032936. \end{aligned}$$

Since $D_h(f)$ is the supremum of the virtual homological spectral radii over the collection of all finite cover, it suffices to construct 1 example of finite type cover of $\Sigma_{1,1}$ for the bound to hold.

3.3 Introduction to the Software

The software works in the following stages:

A. Software prompts for the user input: the size of n to generate the group S_n, and the generators of the fundamental group of the base space, $\Sigma_{1,1}$

- The integer n will be used to construct all the possible finite n-sheeted covering graphs using monodromy representation
- The generators of $\pi_1(\Sigma_{1,1})$, will each correspond to 1 edge in $\Sigma_{1,1}$

B. The software generates $\binom{n!}{r}$ covering graphs, where r is the number of generators of $\pi_1(\Sigma_{1,1})$.

- Note that this includes covering graphs which are also not connected. Thus, after generating the covers, the software sieves through all the covers and discards the disconnected covers.

C. Next, the software computes the fundamental group of the covering graph, $\widetilde{\Sigma_{1,1}}$.

- This is done through using theorems from Combinatorial Group Theory. This step is important to check for which covers the pseudo-Anosov function f lifts.

D. We check for for which covers the pseudo-Anosov function f lifts, using standard results from Algebraic Topology.

E. Once we have trimmed the collection of covers to those which are connected and for which f lifts, we compute the homological representation using a variation of Fox Derivatives.

F. From the homological representations, we compute the necessay eigenvalues and verify the bounds.

3.4 Organisation and Strategy of the Paper

The organisation of the paper generally follows that of the software. The first couple of chapters are purely expository. They are added to provide the background in this topic. In the remaining chapters, we provide an introduction to relevant theories vital to the construction of the software. After each section, the computational steps motivated by the chapter will be worked out in detail; we will work out the calculations step by step for the specific case, $f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on 4-sheeted covers of $\Sigma_{1,1}$.

Once, we have theoretically developed the construction of the software and showcased the case mentioned in the previous paragraph, I will show the results and explain how for the cases we have considered, the bound holds. Lastly, I will explain how I have minimized the degree of error by adding in checks in the algorithm and will suggest on improvements that can be made in the future.

4 Mapping Class Groups

4.1 Introduction to Mapping Class Groups

Let $\Sigma_{g,n}$ be the connected sum of $g \geq 0$ tori with $b \geq 0$ disjoint open disks removed and $n \geq 0$ points removed from the interior. Let $\text{Homeo}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))$ represent the group of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ that restrict to the identity on the boundary of $\Sigma_{g,n}$, $\delta(\Sigma_{g,n})$. We give this group with the compact-open topology.

Definition 4.1 (Compact-Open Topology)**.** *Let X and Y be two topological spaces, and let $C(X, Y)$ denote the set of all continuous maps between X and Y. Given a compact subset K of X and an open subset U of Y, let $V(K, U)$ denote the set of all functions, $f \in C(X, Y)$ such that $f(K) \subset U$. Then the collection of all such $V(K, U)$ is a subbase for the compact-open topology on $C(X, Y)$.*

The mapping class group of surface $\Sigma_{g,n}$, denoted as $\text{Mod}(\Sigma_{g,n})$ is the group:

$$\text{Mod}(\Sigma_{g,n}) = \pi_0(\text{Homeo}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))).$$

This means that we have base point preserving maps from the 0-dimensional sphere (with a given base point) into the space of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ that restrict to the identity on the boundary $\delta(\Sigma_{g,n})$ and these maps from the 0-dimensional sphere to the homeomorphsms are collected into equivalence classes called homotopy classes. Meaning, $\text{Mod}(\Sigma_{g,n})$ is the group of isotopy classes of elements of $\text{Homeo}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))$, where the isotopies fix the boundary pointwise. If $\text{Homeo}_0(\Sigma_{g,n}, \delta(\Sigma_{g,n}))$ denotes the connected component of the identity in $\text{Homeo}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))$, then equivalently we can define $\text{Mod}(\Sigma_{g,n})$ as:

$$\text{Mod}(\Sigma_{g,n}) = \frac{\text{Homeo}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))}{\text{Homeo}_0(\Sigma_{g,n}, \delta(\Sigma_{g,n}))}.$$

Namely, it is the group of orientation-preserving homeomorphism modulo the relation of isotopy.

Other than this definition, there are numerous variations in the definition of $\text{Mod}(\Sigma_{g,n})$, which still results in the equivalent groups. In one such variation, mapping class groups can be defined with respect to diffeomorphisms instead of homeomorphisms. It is also possible to define mapping class groups with repsect to homotopy classes instead of isotopy classes. These definitions would result in isomorphic groups. In essence:

$$\begin{aligned} Mod(\Sigma_{g,n}) &= \pi_0(\text{Homeo}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))) \\ &\approx \text{Homeo}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))/ \text{ homotopy} \\ &\approx \pi_0(\text{Diff}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))) \\ &\approx \text{Diff}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))/\sim, \end{aligned}$$

where $\text{Diff}^+(\Sigma_{g,n}, \delta(\Sigma_{g,n}))$ is the group of orientation-preserving diffeomorphims of $\Sigma_{g,n}$, that are the identity on the boundary and $\sim$ can be taken to be either smooth homotopy relative to the boundary or smooth isotopy relative to the boundary.

There are a few things we need to take note of:

Punctures vs Marked points. Let $\Sigma_{g,n}$ be a surface with $n > 0$ punctures. Then, at certain times, it is more convenient to think of puncutures as marked points. Then, the $\text{Mod}(\Sigma_{g,n})$ is the group of homeomorphisms of $\Sigma_{g,n}$ that leave the set of marked points invariant, modulo isotopies that leave the set of marked points invariant. Note that they are invariant as a set, thus we can permute the marked points among themselves. However, when using homotopies instead of isotopies, we need to be watchful of the following point: a homotopy of surfaces with marked points must not only send marked points to marked points at all times but also must send ummarked points to unmarked points at all times.

Boundaries vs Punctures. By the definitions we give the mapping class group, a key difference between a surface with punctures and a surface with boundary is that, mapping classes are allowed to permute punctures on the surface (they are invariant as sets) but it must preserve the individual boundary components pointwise. The same goes for isotopies: they must fix each boundary pointwise, but they can rotate a neighborhood of a puncture.

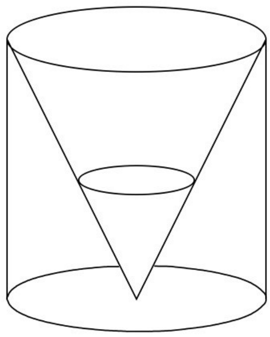
Exceptional Surfaces Note that for 4 surfaces called exceptional surfaces: the disk $\mathbb{D}^2$, the annulus A, the once-punctured sphere $S_{0,1}$, and the twice-punctured sphere $S_{0,2}$, homotopy is not the same as isotopy unless we are dealing with the case of orientation-preserving homeomorphisms. Thus, even when dealing with exceptional surfaces, the variations in definitions of the mapping class group of these surfaces still result in equivalent groups.

We will now calculate the mapping class groups of one of the simplest surfaces, the closed disk $\mathbb{D}^2 \subset \mathbb{R}^2$, using just the definition of the mapping class group.

Theorem 4.1. *(The Alexander Lemma) The group $Mod(\mathbb{D}^2)$ is trivial.*

Proof. Let $\phi : \mathbb{D}^2 \to \mathbb{D}^2$ be an arbitrary homeomorphism that fixes the boundary pointwise. We want to show that for any given homeomorphism of $\mathbb{D}^2$, there exist an isotopy of the homeomorphism to the identity homeomorphism on the disk, $\mathbb{D}^2$.

Intuitively, at time $t = 0$, we want the ϕ to act on $\mathbb{D}^2$. At time $t = s$, where $0 < s < 1$, we want ϕ to act on only the disk with radius $1 - s$ which is contained within $\mathbb{D}^2$. The remaining region of $\mathbb{D}^2$ will remained fixed. Thus, The annulus from radius $1 - s$ to 1 will remained fixed. This can be visualised by the diagram below:



The top of the cylinder denotes time $t = 0$. Thus, ϕ acts on the entire $\mathbb{D}^2$. As we go down cylinder at time $t = s$, ϕ acts on the radius $1 - s$, while the remaining region of $\mathbb{D}^2$ is fixed. This is depicted by the inverted cone inside the cylinder. Then, we can see at $t = 1$, the entire $\mathbb{D}^2$ is fixed, is the function acting on $\mathbb{D}^2$ is the identity homeomorphism.

Thus, we define the homotopy from ϕ to the identity homeomorphism as:

$$F(x,t) = \begin{cases} (1-t)\phi(\frac{x}{1-t}) & 0 \leq |x| < 1-t \\ x & 1-t \leq |x| \leq 1 \end{cases}$$

for $0 \leq t < 1$ and we define $F(x, 1)$ to be the identity map of $\mathbb{D}^2$. This provides an isotopy F from ϕ to the identity, thus proving the statement.

■

Theorem 4.2. *(Mapping Class Group of the Torus) The homomorphism*

$$\sigma : Mod(T^2) \to \mathbb{SL}(2, \mathbb{Z})$$

given by the action on $H_1(\mathbb{T}^2; \mathbb{Z}) \cong \mathbb{Z}^2$.

Proof. Any homeomorphism ϕ of $\mathbb{T}^2$ gives the map $\phi_* : H_1(\mathbb{T}^2, \mathbb{Z}) \to H_1(\mathbb{T}^2, \mathbb{Z})$. As ϕ is invertible, ϕ_* is an automorphism of $H_1(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}^2$. Homotopic maps induce the same map on homology, and so the map $\phi \to \phi_*$ generates the map $\sigma : \text{Mod}(\mathbb{T}^2) \to \text{Aut}(\mathbb{Z}^2 \cong \mathbb{GL}(2, \mathbb{Z}))$. $\sigma(f)$ is an element of $\mathbb{SL}(2, \mathbb{Z})$ as we know that the algebraic intersection numbers in $\mathbb{T}^2$ correspond to determinants and orientation preserving homeomorphism preserve algebraic intersection number.

Next, we need to show that σ is surjective. Any element M of $\mathbb{SL}(2,\mathbb{Z})$ gives an orientation preserving linear homeomorphism of $\mathbb{R}^2$ which is equivariant with respect to the deck transformation group $\mathbb{Z}^2$. Therefore, it descends to a linear homeomorphism ϕ_M of the torus $\mathbb{T}^2 = \frac{\mathbb{R}^2}{\mathbb{Z}^2}$. As we identified primitive vectors in $\mathbb{Z}^2$ with homotopy classes of oriented simple closed curves in $\mathbb{T}^2$, it follows that $\sigma([\phi_M]) = M$, and so σ is surjective.

Now, we show that σ is injective. As $\mathbb{T}^2$ is a $K(G,1)-$space, there exists a correspondence:

$$\{\text{Homotopy classes of based maps } \mathbb{T}^2 \to \mathbb{T}^2\} \longleftrightarrow \{\text{Homomorphisms } \mathbb{Z}^2 \to \mathbb{Z}^2\} \quad (1)$$

Moreover, any element f of $\text{Mod}(\mathbb{T}^2)$ has a representative ϕ that fixes a basepoint for $\mathbb{T}^2$. So, if $f \in ker(\sigma)$, then ϕ is homotopic (as a based map) to the identity, and it follows that σ is injective. In fact, it is possible to create the homotopy of ϕ to the identity explicitly. Similar of that of the annulus, the straight-line homotopy between the identity map of $\mathbb{R}^2$ and any lift of ϕ is equivariant and thus descends to a homotopy between ϕ and the identity.

■

Surprisingly, it was discovered that the mapping class group for any surface can be generated by a composition of twists on any set of simple, closed curves that fill the surface. These twists, which are called Dehn twists, can be viewed as the building blocks of mapping class groups. Dehn twist play the role for mapping class groups that elementary matrices play for linear groups. This can be seen from the Dehn-Lickorish Theorem, which states that the $\text{Mod}(\Sigma_{g,n})$ is generated by finitely many Dehn twists about non-separating simple closed curves, will be eventually seen a couple of sections below. Before that, I will introduce the concept of intersection numbers and Dehn twists.

4.2 Dehn Twists

Now, we will define Dehn twists. This twist was introduced by Max Dehn, who originally called it the screw map. Dehn twists play a central role in the theory of mapping class group, which we will see later on. Before I start on Dehn twists, we need to introduce a couple of preliminary concepts.

There are 2 natural ways to count the number of intersection points between 2 simple closed curves in a surface which correspond to the algebraic intersection number and geometric intersection number respectively: : sign and unsigned. The most intuitive way to count the intersections between the homotopy classes of closed curves is to count the minimal number of unsigned intersections (unsigned being we do not take orientation of the 2 curves into account). This idea is called the geometric intersection number. Formally, the geometric intersection number between free homotopy classes a and b of simple closed curves in a surface $\Sigma_{g,n}$ is defined to be the minimal number of intersection points between a representative curve in the class a and a representative curve in the class b:

$$i(a, b) = min\{|a \cap b| : \alpha \in a, \beta \in b\}.$$

Readers interested in the algebraic intersection numbers can read more about it at [Mar12].

Now, let's consider the annulus $A = S^1$ x $[0, 1]$. To orient A, we embed it in the (θ, r)-plane using the function:

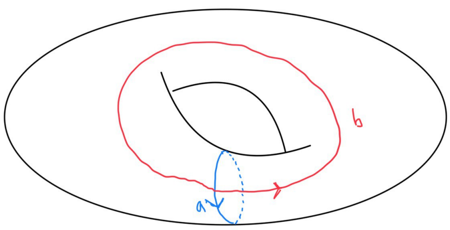
$$(\theta, t) \rightarrow (\theta, t + 1)$$

and the orientation is given by the standard orientation of the (θ, r)-plane.

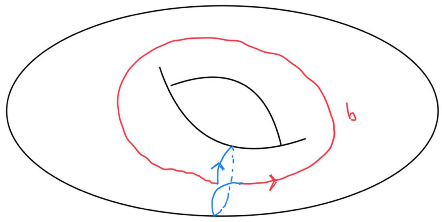
Let $T : A \rightarrow A$ be the twist map of A:

$$T(\theta, t) = (\theta + 2\pi t, t).$$

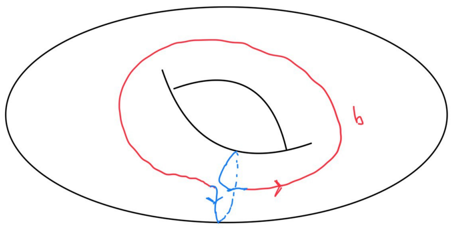
The map T is an orientation-preserving homeomorphism that fixes the boundary of the annulus, $\delta(A)$ pointwise. Note that instead of using $\theta + 2\pi t$ we could use $\theta - 2\pi t$. The first choice is a left twist, while the other is a right twist.
To visualise the difference, let's take a look at the simple case of 2-torus, $\Sigma_{1,0}$. For a better intuition, we can think of this as a person walking along the curves on $\Sigma_{1,0}$. As shown below (and proved later), let a and b be the set of simple closed curves on $\Sigma_{1,0}$ that fills it.



Then, a left twist on curve a results in a person walking along curve b. At the point of intersection of curves a and b, then he turns left and walks along a. After he finishes walking along curve a, he resumes walking on b, till he reaches his original point. Note that in the left-twist, he walks against the original direction of a. This can be seen below:



For a right twist on curve a, the person walks along curve b and at the point of intersection of curves a and b, then he turns right and walks along a. After he finishes walking along curve a, he resumes walking on b, till he reaches his original point. Note that in the right-twist, he walks against the original direction of a. This can be seen below:



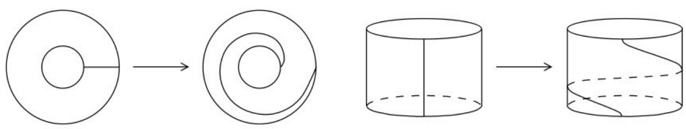
So, the $\pm$ distinguishes the orientation of the twist.

Now, I will give the definition of the Dehn twist. Let $\Sigma_{g,n}$ be an arbitrary (oriented) surface and let α be a simple closed curve in $\Sigma_{g,n}$. Let N be a regular neighborhood of α and choose an orientation-preserving homeomorphism $\phi : A \to N$. We obtain a homeomorphism $T_\alpha : \Sigma_{g,n} \to \Sigma_{g,n}$, called a Dehn twist about α, as follows:

$$T_\alpha(x) = \begin{cases} \phi \cdot T \cdot \phi^{(-1)}(x) & \text{if } x \in N \\ x & \text{if } x \text{ in } \Sigma_{g,n} \text{ but not in } N. \end{cases} \tag{2}$$

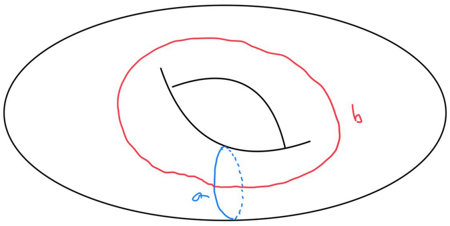
Intuitively, this just means that for T_α, we "perform the twist map T on the annulus A and fix every point outside of N."

The Dehn twist can be viewed accordingly in 2 ways:

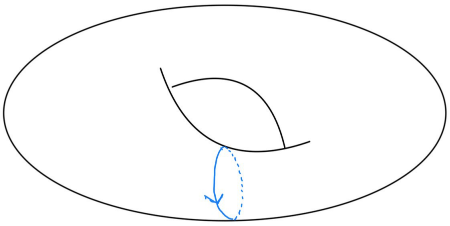
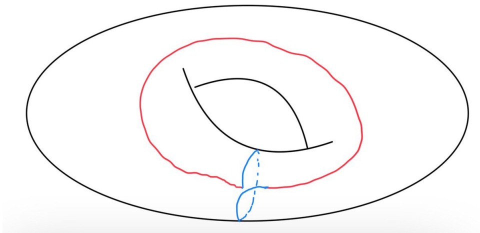


Note that if the 2 curves α_1 and α_2 are isotopic, then T_{α_1} is isotopic to T_{α_2} as well. That is to say, they are well defined elements in $\mathrm{Mod}(\Sigma_{g,n})$.

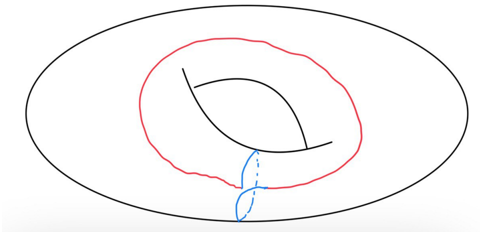
Dehn twists on the torus Let's take a look at how the Dehn twist on the curves a and b look like on the torus.



When we perform a twist on a on the torus, we can see that curve a is sent to itself (figure on the left below) and curve b loops around curve a and b (figure on the right below).

When we perform a twist on b on the torus, we can see that curve b is sent to itself (figure on the right below) and curve a loops around curve a and b (figure on the left below).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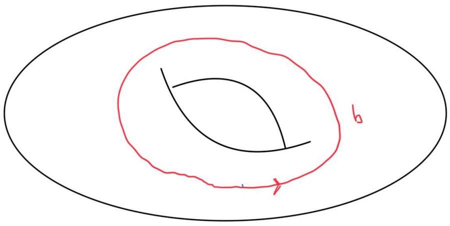
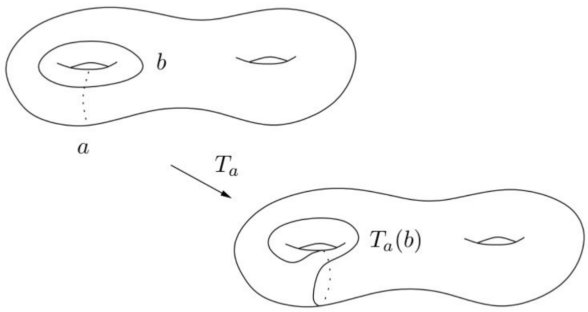


Dehn twists via Cutting and gluing Another way to think about dehn twists T_α is as follows: We can cut $\Sigma_{g,n}$ along α, twist the neighborhood of one boundary component through an angle of 2π, and the re-glue along the cut components. These steps gives a well-defined homeomorphism of S which is equivalent to T_α.



Action on Simple Closed Curves Simple curves are curves that does not intersect itself. By observing its action on the isotopy classes of simple closed curves on $\Sigma_{g,n}$, we can understand T_α further.

- If $i(a,b) = 0$, then $T_a(b) = b$.
- Else, if $i(a,b) \neq 0$, the isotopy class of $T_a(b)$ is determined as follows: given s particular representatives β and α of b and a, each segment of β intersecting α is replaced with a segment that turns left, follows α around, and then turns right. This is true no matter which way we orientate β.

Left vs Right Once an orientation of $\Sigma_{g,n}$ is fixed, the direction of a twist T_a does not depend on the orientation on a. This is because turning left is well-defined on an oriented surface. The inverse map $T_a^{-1}$is simply a twist about a on the other direction; it is similarly defined to T_a, with the twist map T replaced by its inverse T^{-1}.

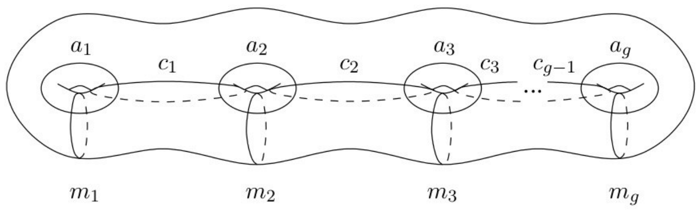
Now, we shall briefly introduce the fact that that the mapping class group is generated by the dehn twist.

4.3 Dehn-Lickorish Theorem

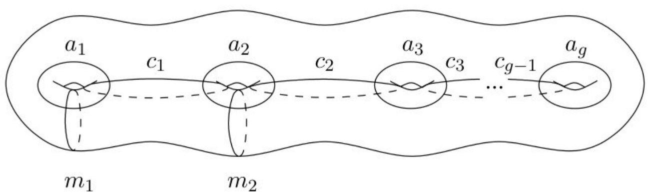
Our main goal in this subsection is to introduce the Dehn-Lickorish Theorem and give a sketch of it's proof. Note that in this subsection, we will denote the surface by S or S_g and the permutation group of size n as Σ_n to avoid any confusion.

Definition 4.2. *(Dehn-Lickorish Theorem) For $g \geq 0$, the mapping class group Mod(S_g) is generated by finitely many Dehn twists about non-separating closed curves.*

In 1938, Dehn proved that $\mathrm{Mod}(S_g)$ is generated by $2g(g-1)$ Dehn twists [Deh10]. Expanding on Dehn's results, Mumford showed in 1967 that only Dehn twists about non-separating curves were needed [Mum67]. Then, in 1964, Lickorish proved that $\mathrm{Mod}(S_g)$ is generated by the Dehn twists about the $3g-1$ non-separating curves as shown below in the picture [Lic64].



In 1979, Humphries [Hum77] proved that the twist about the $2g+1$ curves in the figure below suffice to generate $\mathrm{Mod}(S_g)$. These generators are often called the Humphries generators.



Punctures and Pure Mapping Class Groups. The Dehn-Lickorish Theorem is not true for surfaces with multiple punctures as no composition of Dehn Twists can permute the punctures. Let $\mathrm{PMCS}(S_{g,n})$ be denote the pure mapping class group of $S_{g,n}$, which is defined for the subgroup of $\mathrm{Mod}(S_{g,n})$ consisting of elements that fix each puncture individually. The action of $\mathrm{Mod}(S_{g,n})$ on the punctures of $S_{g,n}$ results in the following short exact sequence:

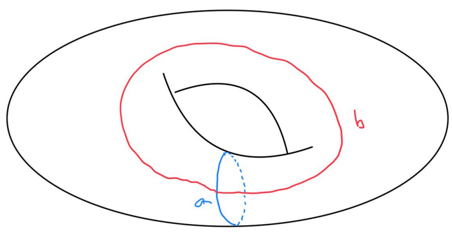
$$1 \to PMod(S_{g,n}) \to Mod(S_{g,n}) \to \Sigma_n \to 1,$$

where Σ_n is the permutation group on the n punctures. Thus, we can see that in the case $n = 1$, $PMCS(S_{g,n}) = MCG(S_{g,n})$.

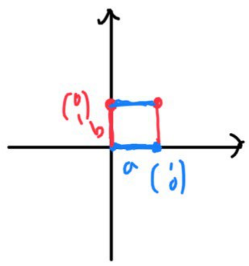
Outline of Proof of Dehn-Lickorish Theorem In proving the Dehn-Lickorish theorem, we need to prove a more general statement: we will prove that $\text{PMod}(S_{g,n})$ is generated by finitely many Dehn Twist about non-separating simple closed surfaces for any $g \geq 1$ and $n \geq 0$. We will first need give a sketch of a weaker statement that $\text{PMod}(S_{g,n})$ is generated by the (infinite) collection of all Dehn Twist about nonseparating simple closed curves. The argument is a double induction on the number of genus ,g, and number of punctures, n, on the surface. The base case will be $\Sigma_{1,1}$. In order to motivate 2 important tools: the complex of curves and the Birman Exact Sequence. This proof is not integral to the rest of this thesis, so interested readers can read more about it in [Mar12].

4.4 Dehn Twist Generating the mapping class group of the once punctured torus

As noted above, when the number of punctures $n = 1$, $PMCS(S_{g,n}) = MCG(S_{g,n})$. Thus, the mapping class group of the once-puncutured torus is exactly the same as the mapping class group of the 2-tours. Thus, the mapping class group of $\Sigma_{1,1}$ is the group $SL_2(\mathbb{Z})$. By the Dehn-Lickorish theorem, we know that there are a finite number of Dehn-Twists generating the mapping class group of the once-puncutured torus. Using the set of generators constructed by Humphries, we can see that the set of Dehn twist that generate $\text{Mod}(\Sigma_{1,1})$ are the Dehn Twists performed on the curves a and b we have seen before:



On $\mathbb{R}$, we can embed the 2-cell representation of the torus like this:



The isomorphism $\text{Mod}(\Sigma_{1,1}) \cong SL_2(\mathbb{Z})$ tells us that there are 2 matrices in $SL_2\mathbb{Z}$, each of which corresponds to the Dehn twist on a and b.

First, note that there is a finite set of matrices generating $SL_2(\mathbb{Z})$.

Theorem 4.3. *The generators of $SL_2(\mathbb{Z})$ are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. [Con]*

Note that the 2 simple closed curves a and b in $\Sigma_{1,1}$ fills the surface. We can view the curve a as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (ie on the x-axis) and the b curve as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (ie on the y-axis). Thus, it turns out that we can express any element of the mapping class group of the once-punctured torus as a composition of the dehn twist on curve a, T_a, and the dehn twist on curve b, T_b:

$$T_a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$T_b \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

By studying the action of T_a and T_b on a and b in $\Sigma_{1,1}$, we can understand how any function $\phi \in SL_2(\mathbb{Z})$ act on $\Sigma_{1,1}$ as ϕ can be expressed as a composition of T_a and T_b.

We can see that generators T_a and T_b acts on $\Sigma_{1,1}$ accordingly:

- Action of T_a on $\Sigma_{1,1}$:
  - $a \mapsto a$
  - $b \mapsto b + a$.
- Action of T_b on $\Sigma_{1,1}$:

- $a \mapsto a + b$
- $b \mapsto b$.

Moreover, we can express the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ as a composition of the dehn twist on curve a, T_a, and the dehn twist on curve b, T_b:

$$A = T_a \cdot T_b$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Thus, we can see that matrix A acts on the space filling curves of T^2 accordingly:

$$a \mapsto a + b + a$$
$$b \mapsto b + a.$$

Expressing the action of A in terms of the generator of the free group a and b, gives us a way to study the matrix A with a computer.

5 Pseudo-Anosov Mapping Classes

Pseudo-Anosov mapping classes are one of the 3 special types of mapping classes. In this section, I will introduce the definition of pseudo-Anosov mapping classes. As these mapping classes are intrinsically linked to the concept of measured singular foliations, I will explain these foliations on the simple case of the torus first, and then generalise to higher genus surfaces.

Once we have established the theory of pseudo-Anosov maps, we will show how Thurston's Construction and Penner's construction can work together to construct pseudo-Anosov mapping classes for any given surface.

For the case of the once-punctured torus $\Sigma_{1,1}$, it is easier to prove if a mapping class $f \in \mathrm{Mod}(\Sigma_{1,1})$, using the Nielsen-Thurston Classification theorem, applied to $\Sigma_{1,1}$ via linear algebra. Thus, I will introduce these and finally prove that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is pseudo-Anosov.

5.1 Definition of Pseudo-Anosov

Definition 5.1. *(Pseudo-Anosov Mapping Classes) An element $f \in Mod(\Sigma_{g,n})$ is called pseudo-Anosov if there is a pair of transverse measured folaitions $(\mathbb{F}^u, \mu_u)$ and $(\mathbb{F}^s, \mu_s)$ on $\Sigma_{g,n}$, a number $\lambda > 1$, and a representative homeomorphism ϕ so that*

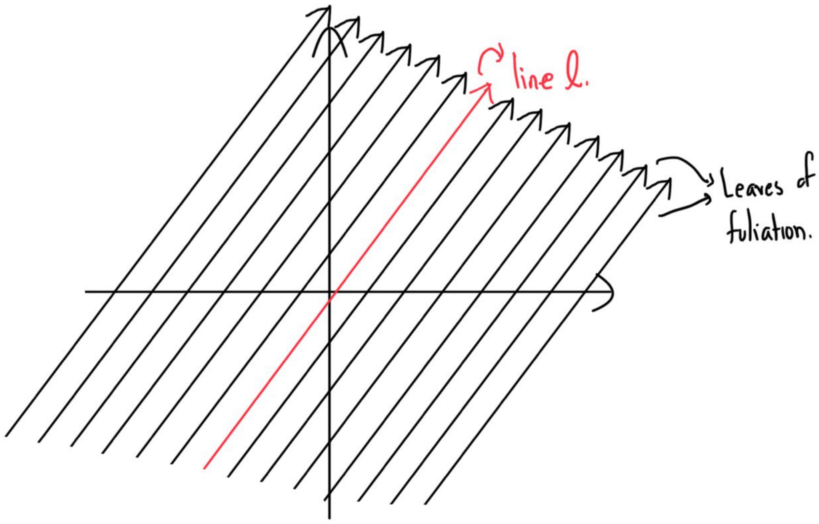
$$\begin{aligned}\phi(\mathbb{F}^u, \mu_u) &= (\mathbb{F}^u, \lambda\mu_u), \ \textit{and} \ , \\ \phi(\mathbb{F}^s, \mu_s) &= (\mathbb{F}^s, \frac{1}{\lambda}\mu_s).\end{aligned}$$

The measured foliations $(\mathbb{F}^u, \mu_u)$ and $(\mathbb{F}^s, \mu_s)$ are called the unstable and stable foliation respectively and the number λ is called the stretch factor of ϕ or of f. Then, the map ϕ is called a pseudo-Anosov homeomorphism.

5.2 Measured Singular Foliation on the Torus

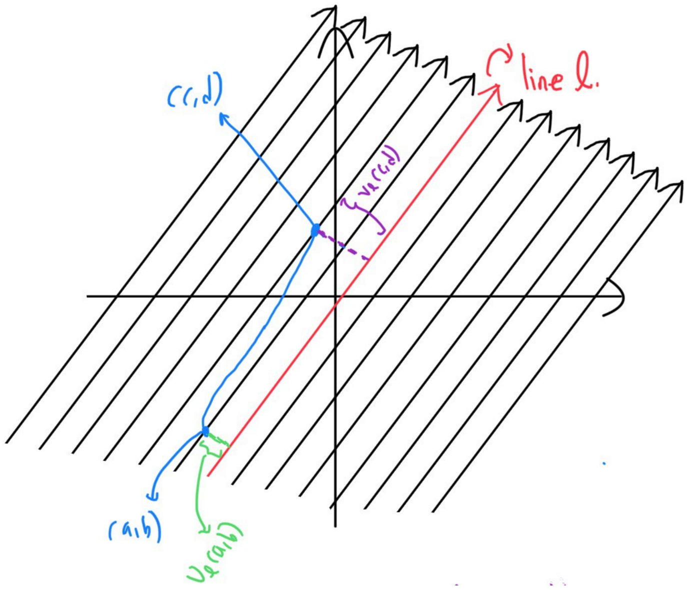
We will first focus our attention on the simple case of the torus before providing the general definition of a measured foliation. I will also elaborate on what it means for a linear map of the torus to stretch the torus along one foliation and shrink along the other.

Let l be any line passing through the origin in $\mathbb{R}^2$. The line l determines a foliation $\widetilde{F_l}$ of $\mathbb{R}^2$, which consists of all lines in $\mathbb{R}^2$ parallel to l. Translations of $\mathbb{R}^2$ takes lines to lines, and so any translation preserves $\widetilde{F}_l$, meaning that leaves are sent to leaves.

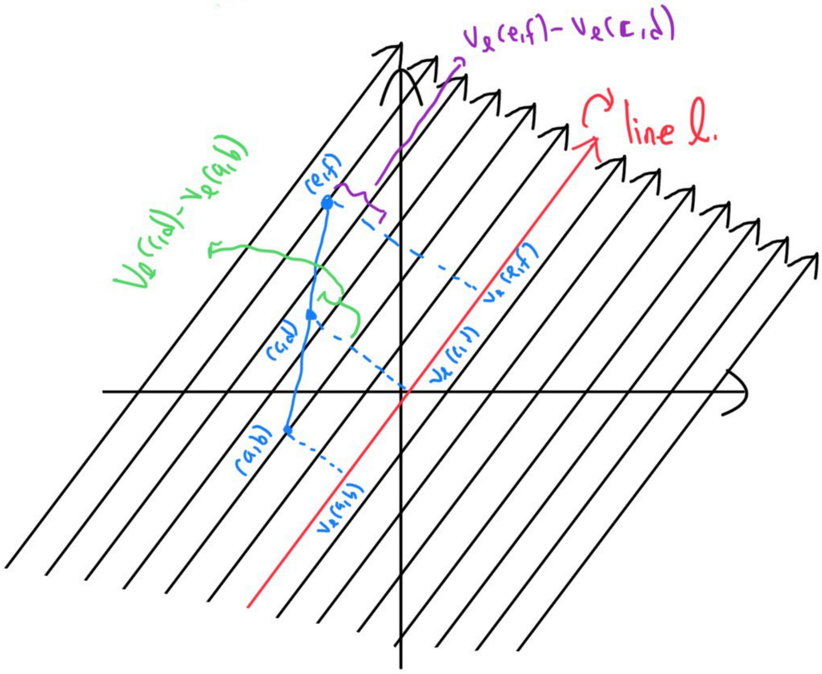


Since all deck transformations for the standard covering $\mathbb{R}^2 \to T^2$ are translations, the foliation $\widetilde{F}_l$ descends to a folation F_l of T^2.

There is an additional structure we will equip the foliations $\widetilde{F}_l$ with. Let $v_l : \mathbb{R}^2 \to \mathbb{R}$ be the function that records distance from any point in $\mathbb{R}^2$ to l. In this picture, the arc from (a, b) to (c, d) denotes a transverse arc in $\mathbb{R}^2$:



v_l will allow us to calculate the difference in height between different points in $\mathbb{R}^2$ with respect to the line l.



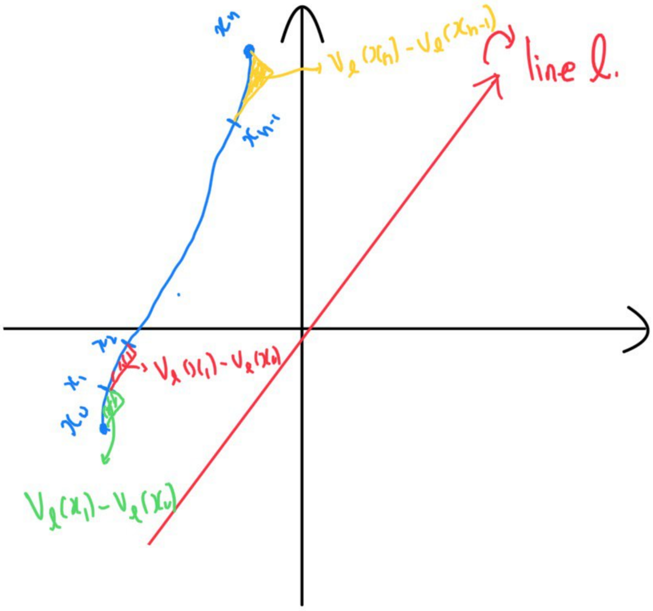
This in turn will let us define a measure in $\mathbb{R}^2$.

The measure μ with respect to the foliation is a function that associates each smooth arcs transverse to foliation F to a real number.

$$\mu : \{\text{Smooth Arcs transverse to foliation } \widetilde{F_l}\} \mapsto \mathbb{R}$$

Integration against the 1-form dv_l provides a transverse measure on $\widetilde{F_l}$. This means that any smooth arc α transverse to the leaves of $\widetilde{F_l}$ can be assigned a length defined by $\mu(\alpha) = \int_\alpha dv_l$. The quantity $\mu(\alpha)$ is the total variation of α in the direction perpendicular to l.

Let's break down what this means. Let $P = \{x_0, x_1, ..., x_n\}$ denote the partition of the smooth arc α transverse to foliation F.



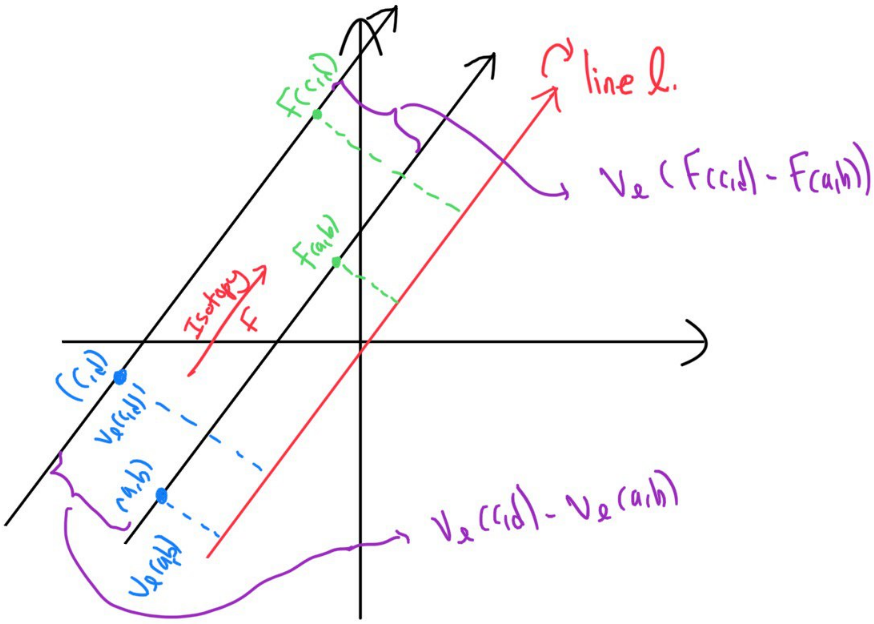
The measure of a smooth arc α transverse to F is defined as shown below:

$$\mu(\alpha) = \sup_P \sum_{i=0}^{n-1} |v_l(x_{i+1}) - v_l(x_i)|,$$

where P is the collection of all possible partitions of α.

So, intuitively, we can see that each smooth arc transverse to the foliation is given a real number which measure the total oscillation from the start point to the end point.

Now, we shall describe some properties of the measure. Firstly, note that $\mu(\alpha)$ is invariant under isotopies of α that move each point α within the leaf of $\widetilde{F}_l$ in which it is contained.



The reason for this is that if a point only shifts within the leave it is contained in, there is no change in height with respect to the base line l (Note that each leaf in the foliation is parallel to l). Since μ can be intepreted as the integration against the 1-form dv_l, this tells us that the measure is indeed invariant under isotopies of α that move each point α within the leaf of $\widetilde{F}_l$ in which it is contained.

Next, The 1-form dv_l is preserved by translations. This is due to similar reasoning above. As the entire plane is shifted by the translation, at if a point only shifts within the leave it is contained in, there is no change in height with respect to the base line l. So, since μ can be interpreted as the integration against the 1-form dv_l, the 1-form dv_l is preserved. So, the 1-form dv_l descends to a 1-form w_l on T^2 and induces a transverse measure on the foliation F_l. The structure of a foliation on T^2 together with a transverse measure is called a transverse measured foliation on T^2.

5.3 Worked out examples of foliation of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

The characteristic polynomial of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is:

$$\lambda^2 - tr(A)\lambda + det(A) = 0.$$

Since $tr(A) = 2 + 1 = 3$ and $A \in SL(2, \mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}$, we have:

$$\lambda^2 - 3\lambda + 1 = 0.$$

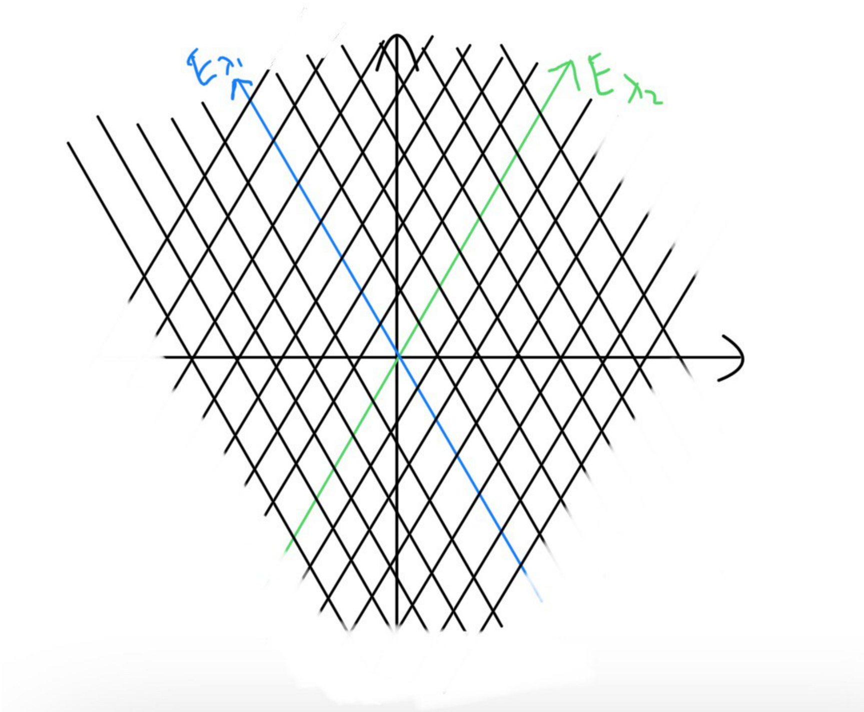
Thus, we have the following eigenvalues:

$$\lambda_1 = \frac{-\sqrt{5} + 1}{2}$$
$$\lambda_2 = \frac{\sqrt{5} + 1}{2}.$$

The corresponding eigenspace of λ_1 and λ_2, denoted as E_{λ_1} and E_{λ_2} respectively, are:

$$E_{\lambda_1} = \{m \begin{bmatrix} \frac{-\sqrt{5}+1}{2} \\ 1 \end{bmatrix} : m \in \mathbb{R}^2\}$$
$$E_{\lambda_2} = \{n \begin{bmatrix} \frac{\sqrt{5}+1}{2} \\ 1 \end{bmatrix} : n \in \mathbb{R}^2\}$$

This results in the foliations on T^2:



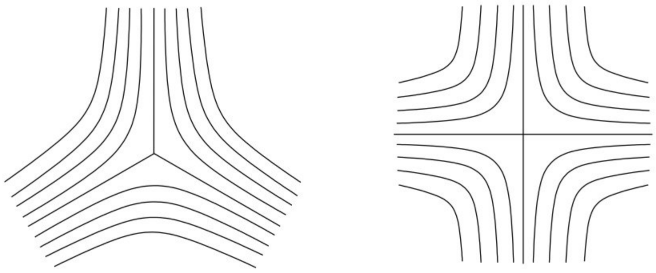
5.4 Measured Singular Foliations on $\Sigma_{g,n}$

On a higher genus surface, it is not clear what it means for a homeomorphism to stretch in the direction of a single vector. To counter this, we can construct measured foliation on a higher genus surface, which we will then map it to $\mathbb{R}^2$ using smooth charts. This will allow us to embed the foliations in $\mathbb{R}^2$, where we can see how a homeomorphism to stretch the surface in the direction of that foliation.

A singular foliation F on a closed surface $\Sigma_{g,n}$ is a decomposition of $\Sigma_{g,n}$ into a disjoint union of subsets of $\Sigma_{g,n}$, called the leaves of F, and a finite set of points of $\Sigma_{g,n}$, called singular points of F, such that the following 2 conditions hold:

A. For each non-singular point $p \in \Sigma_{g,n}$, there is a smooth chart from a neighborhood of p to R^2 that takes leaves to horizontal line segments. The transition maps between any 2 of these charts are smooth maps of the form $(x, y) \mapsto (f(x, y), g(y))$. In other words, the transition maps take horizontal lines to horizontal lines.

B. For singular points $p \in \Sigma_{g,n}$, there is a smooth chart from a neighborhood of p to $\mathbb{R}^2$ that takes leaves to level sets of a k-pronged saddle, $k \geq 3$.



Just like how we gave foliations on the torus a measure, we want the foliations on higher genus surfaces to be equipped with the transverse measure too. Which is to say a length function defined on arcs transverse to the foliation. However, first, we need to define leaf-preserving isotopies first.

Let F be a foliation on a surface $\Sigma_{g,n}$. A smooth arc α in $\Sigma_{g,n}$ is transverse to F if α misses the singular points of F and is transverse to each leaf of F at each of its interior point. Let $\alpha, \beta : \mathbb{I} \to \Sigma_{g,n}$ be smooth arcs transverse to F. A leaf preserving isotopy from α to β is a map $H : \mathbb{I} \text{ x } \mathbb{I} \to \Sigma_{g,n}$ such that:

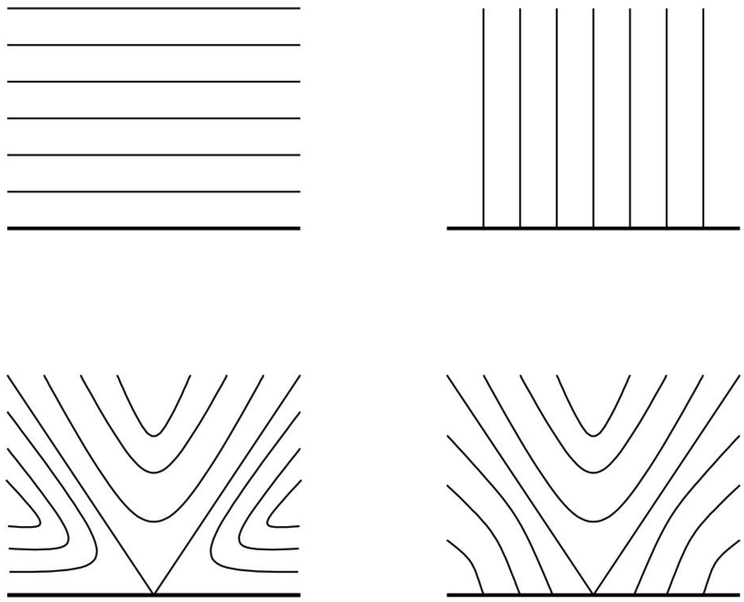
- $H(\mathbb{I} \text{ x } \{0\} = \alpha)$ and $H(\mathbb{I} \text{ x } \{1\} = \beta)$
- $H(\mathbb{I} \text{ x } \{t\}$ is transverse to F for each $t \in [0, 1]$.
- $H(\{0\} \text{ x } \mathbb{I}$ and $H(\{1\} \text{ x } \mathbb{I}$ are each contained in a single leaf.

A transverse measure μ on a foliation F is a map that assigns a positive real number to each smooth arc transverse to F, so that μ is invariant under leaf-preserving isotopy and μ is regular with respect to Lebesgue measure. This means that each point of $\Sigma_{g,n}$ has a neighborhood U and a smooth chart $U \to \mathbb{R}^2$ so that the measure μ is induced by $|dy|$ on $\mathbb{R}^2$.

Thus, a measured foliation (F, μ) on a surface S is a foliation F of S equipped with a transverse measure μ.

Punctures and Boundary At a puncture, a foliation takes the form of a regular point or a k-pronged singularity with $k \geq 3$, as in the case of foliations on closed surfaces. At a puncture, however, we can allow a one prong singularity.

A measured foliation on a compact surface S with nonempty boundary is defined similarly to the case when S is closed. There are four different pictures in the neighborhood of a point in the boundary of S depending on whether or not the point is singular and whether or not the leaves are parallel to the boundary or transverse to the boundary.



Now, that we have defined measured singular foliations, we will need to study on how to construct these mapping classes for a given surface $\Sigma_{g,n}$ so that we can test the relations by Dr Thang Le.

There are primarily 5 constructions of the pseudo-Anosov functions:

A. Branched Covers

B. Dehn Twists Constructions

C. Homological Criterion

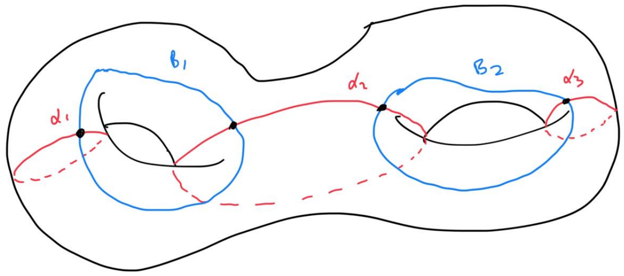
D. Kra's Construction

E. A Construction for Braid Groups

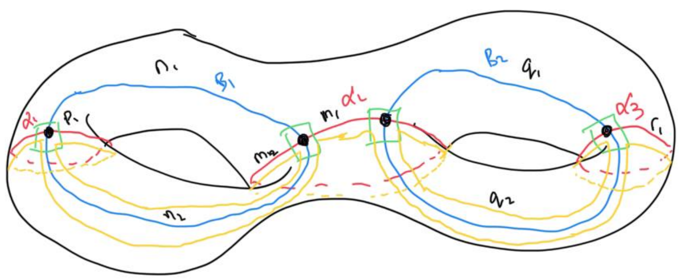
We will focus on the construction of pseudo-Anosov by Dehn twists.

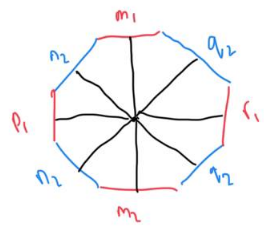
5.5 Construction of pseudo-Anosov via Dehn Twists

In this subsection, we will use the surface $\Sigma_{2,0}$ as a reference to understand the construction of pseudo-Anosovs via Dehn twists. The red and blue curves form the set of multicurves $T_A = \{\alpha_1, \alpha_2, \alpha_3\}$ and $T_B = \{\beta_1, \beta_2\}$. The set of curves $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\}$ fills $\Sigma_{2,0}$.



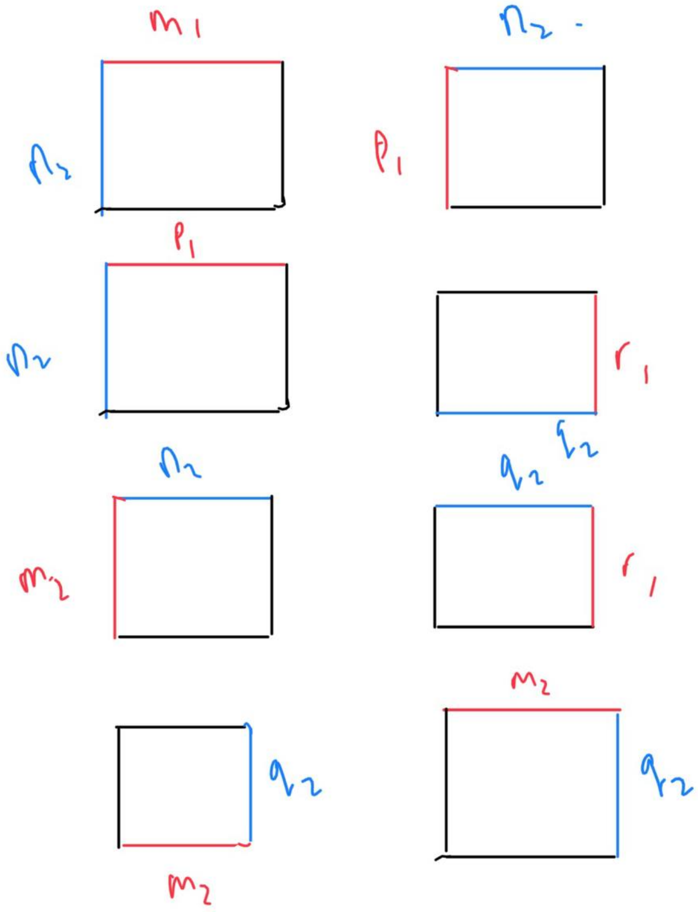
We can think of $\alpha_i \cup \beta_j$ as a 4-valent graph in $\Sigma_{2,0}$, where the vertices are the points of $\alpha \cap \beta$ (represented by the black vertices above in the diagram), as each vertex has degree 4. In fact, by also considering the closures of the components of $S - \alpha \cup \beta$ as 2-cells, we have a description of S as a 2-complex X. By cutting the surface along the 1-complex $T_A \cup T_B$, we can decompose $\Sigma_{2,0}$ into 2 2-cell complexes. One of it is shown below as an example.



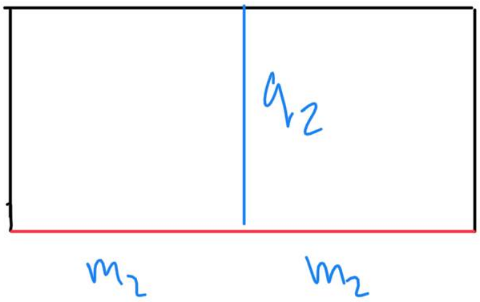


We construct a dual complex X' of $\Sigma_{2,0}$ using the 2 2-cell complexes . This complex is formed by taking one vertex for each 2-cell of X, called the co-vertex, one edge transverse to each edge of X, called co-edges, and one 2-cell for each vertex of X, called co-cells. If the 2-cell has a puncture or a marked point in it, then the marked point/puncture will be the co-vertex. This is also shown above.

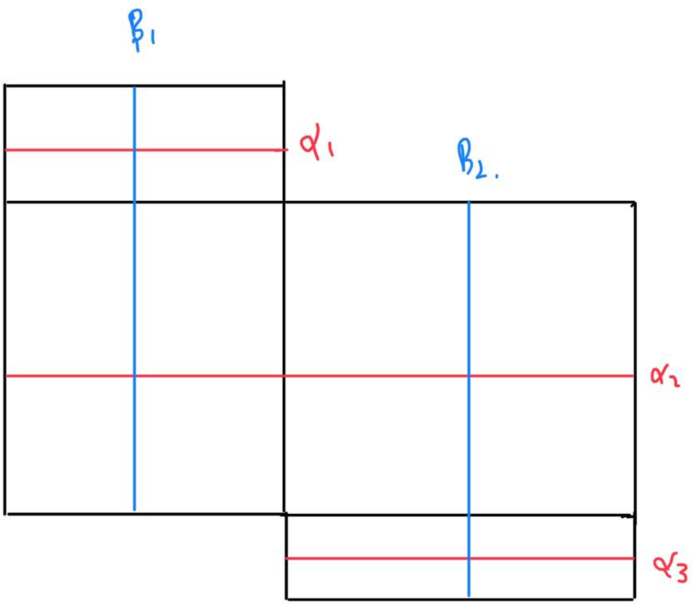
In our case, as our 2-cell is an octagon with 8 edges protruding from the co-vertex, we further decompose our 2 co-cells into 16 rectangles. The case for the co-cell in the above picture is shown below.



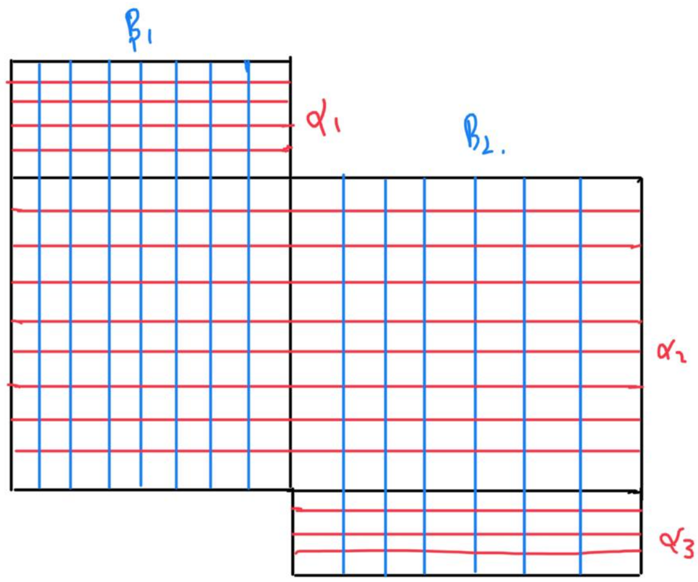
We can glue these squares together to form a bunch of rectangles which can be embedded in $\mathbb{R}^2$.



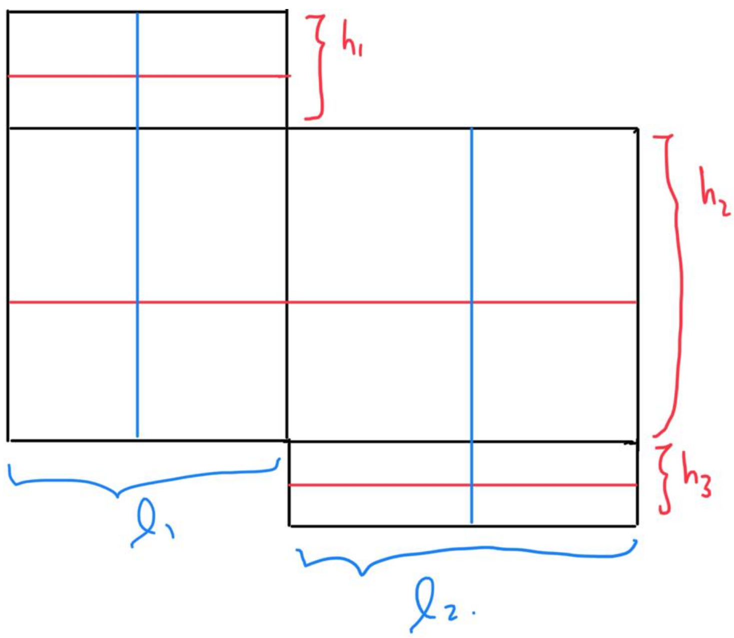
Since the vertices of X are 4-valent, it follows that X' is a square complex, that is each 2-cell of X' is a square. What is more, each square of X' has a segment of α running from one side to the opposite side.



We can foliate each square of X' by lines parallel to α. This gives rise to a foliation F_α on all of S.



We declare the width of each square to be the same fixed number, and this gives a measure on F_α. The foliation associated to β is a measured foliation F_β that is transverse to F_α.



5.6 Thurston Construction on $\Sigma_{2,0}$

Before we give the general statement of Thurston's Construction on any surface, let's consider the much simpler case of $\Sigma_{2,0}$. The essence of the proof of the general statement is similar and can be abstracted from the case of $\Sigma_{2,0}$.

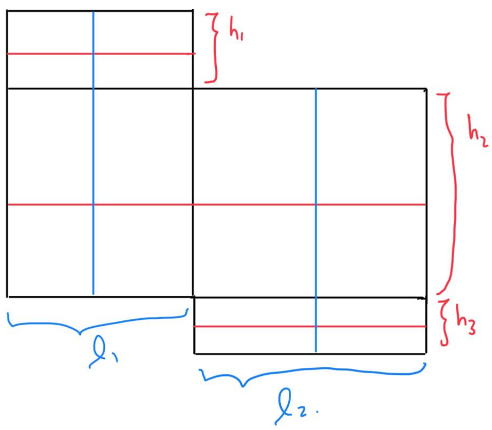
Let A and B be multicurves in $\Sigma_{2,0}$ such that $A \cup B$ fills $\Sigma_{2,0}$. T_A and T_B will denote the multitwist of A and B respectively. Thurston's construction tells us that any mapping class f generated as a composition of T_A and T_B can be classified as periodic, reducible or pseudo-Anosov based on the representation

$$\rho :< T_A, T_B > \to PSL(2, \mathbb{R})$$
$$T_A \mapsto \begin{pmatrix} 1 & -\mu^{\frac{1}{2}} \\ 0 & 1 \end{pmatrix}$$
$$T_B \mapsto \begin{pmatrix} 1 & 0 \\ \mu^{\frac{1}{2}} & 1 \end{pmatrix}$$

An element $f \in < T_A, T_B >$ is periodic, reducible or pseudo-Anosov according to whether $\rho(f)$ is elliptic, parabolic or hyperbolic.

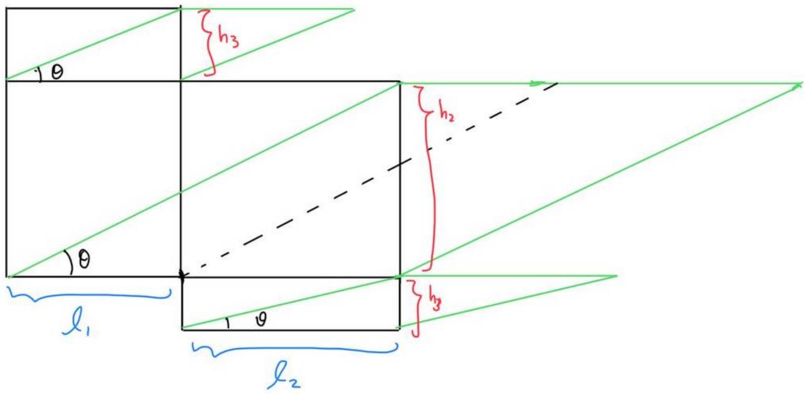
That is to say, by embedding $\Sigma_{2,0}$ in the euclidean plane $\mathbb{R}^2$ and studying how $\rho(f)$ acts on the embedded $\Sigma_{2,0}$, we can figure out the classification of f.

Embedding of $\Sigma_{2,0}$ into $\mathbb{R}^2$: The first step of constructing the representation in Thurston's construction involves embedding $\Sigma_{2,0}$ into $\mathbb{R}^2$. While half the work was done in the previous section decomposing $\Sigma_{2,0}$ into a flat structure (rectangles), we have yet to give geometry to these rectangles. That is to say we have to specify the lengths of h_1, h_2, h_3, l_1 and l_2:

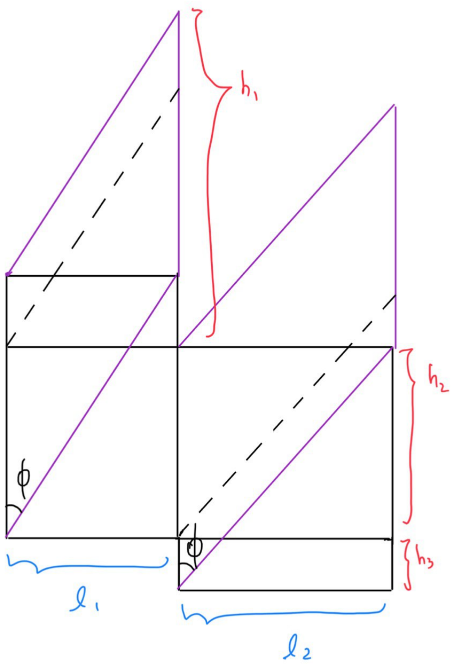


As we will see further down, this boils down to solving a system of linear equations.

Note that as T_A is a Dehn twist on non-intersecting curves $\alpha_1, \alpha_2, and \alpha_3$, the action of T_A on the rectangle is as follows:



Note that as T_B is a Dehn twist on non-intersecting curves β_1 and β_2, the action of T_B on the rectangle is as follows:



2 key facts that will give us the system of linear equations to solve are:

- We want T_A and T_B to act affinely on the rectangles in $\mathbb{R}^2$. This means that we want parallel lines to remain parallel after the transformation.
- T_A and T_B act affinely if and only if the slopes of the rectangles are constant after T_A and T_B act affinely on the rectangles.

This results in the following sets of equations:

$$\lambda = tan(\theta) = \frac{h_1}{l_1} = \frac{h_2}{l_1 + l_2} = \frac{h_3}{l_2}$$
$$\mu = tan(\phi) = \frac{l_1}{h_1 + h_2} = \frac{l_2}{h_2 + h_3}$$

Using the fact that $v = \mu\lambda$, we get the following pair of system of linear equations, one dealing with variables h_i and the other dealing with l_j.

The length equations:

$$\frac{l_1}{v} = 2l_1 + l_2$$
$$\frac{l_2}{v} = l_1 + 2l_2$$

The height equations:

$$\frac{h_1}{v} = h_1 + h_2$$
$$\frac{h_2}{v} = h_1 + 2h_2 + h_3$$
$$\frac{h_3}{v} = h_2 + h_3$$

When the equations above are represented in linear algebra form:

$$\frac{1}{v}\begin{bmatrix} l_1 \\ 1_2 \end{bmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{bmatrix} l_1 \\ 1_2 \end{bmatrix}$$
$$\frac{1}{v}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

Then, we can see that the eigenvectors of the equations above, give us the dimensions for the sides of the rectangles. After obtaining the lengths and the height we can embed $\Sigma_{2,0}$ into $\mathbb{R}^2$.

Constructing the matrix representations of T_A and T_B The μ in the representation in Thurston's Construction comes from the eigenvalue of the equations above.

Generalising the argument above: While we can reduce the problem in the above scenario to simple sets of system of linear equation, this might not always be easy. However, there is an important fact that allows us to skip the process of obtaining the length and height equations via trigonometry and immediately obtain the matrix form: It can be shown that the choice for h_i and l_j can be obtained from the following matrix:

$$N = \begin{pmatrix} i(\alpha_1, \beta_1) & i(\alpha_1, \beta_2) \\ i(\alpha_2, \beta_1) & i(\alpha_2, \beta_2) \\ i(\alpha_3, \beta_1) & i(\alpha_3, \beta_2) \end{pmatrix}.$$

To verify, note that:

$$NN^t = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ and}$$
$$N^tN = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Thus, for the general case with the multicurves $A = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $B = \{\beta_1, \beta_2, ..., \beta_n\}$. Let N be the matrix with (j, k) entry, we define the matrix N as

$$N_{j,k} = i(\alpha_j, \beta_k).$$

Theorem 5.1. *N is primitive.*

Proof. Given N, let G be the abstract bipartite graph with m red vertices and n blue vertices, and $N_{j,k}$ edges between the jth vertex and the kth blue vertex. Then, the (j, k) entry of the dth power $(NN^t)^d$ is equal to the number of paths in G of length $2d$ between the jth and kth red vertices in G. Indeed, this is equivalent to the statement that the graph G is connected. If G is not connected, that would mean that $A \cup B$ is not connected, and so the pair A, B does not fill the surface. Thus, N is primitive.

■

Then, we can use the perron-frobenius theorem:

Theorem 5.2 (Perron-Frobenius matrices). *Let A be an nxn matrix with integer entries. If A is a primitive, then A has a unique nonnegative unit eigenvector v. The vector v is positive and has a positive eigenvalue that is larger in absolute value than all other eigenvalues.*

which tells us we can find the vectors we require.

The general statement of Thurston's construction is below.

5.7 Thurston's Construction

Theorem 5.3 (Thurston's Construction). *Suppose A and B are multicurves in S, so that $A \cup B$ fills S. There is a real number $\mu = \mu(A, B)$ and a representation $\rho :< T_A, T_B >\to PSL(2, \mathbb{R})$ given by:*

$$T_A \mapsto \begin{pmatrix} 1 & -\mu^{\frac{1}{2}} \\ 0 & 1 \end{pmatrix} \text{ and } T_A \mapsto \begin{pmatrix} 1 & 0 \\ \mu^{\frac{1}{2}} & 1 \end{pmatrix}.$$

The representation ρ has the following properties:

- A. *An element $f \in< T_A, T_B >$ is periodic, reducibleor pseudo-Anosov according to whether $\rho(f)$ is elliptic, parabolic or hyperbolic.*
- B. *When $\rho(f)$ is parabolic, f is a multitwist.*
- C. *When $\rho(f)$ is hyperbolic, the stretch factor of the pseudo-Anosov mapping class f is equal to the larger of the 2 eigenvalues of $\rho(f)$.*

5.8 Penner's Construction

By utilising Thurston's Construction, Penner provides us a method to construction pseudo-Anosov maps.

Theorem 5.4 (Penner's Construction). *Let $A = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $B = \{\beta_1, \beta_2, ..., \beta_n\}$ be multicurves in a surface $\Sigma_{g,n}$ that together fill $\Sigma_{g,n}$. Any product of positive powers of the T_{α_i} and negative powers of T_{β_i}, where each α_i and each β_i appear at least once, is pseudo-Anosov.*

In the statement above, order does not matter. Penner has conjectured that every pseudo-Anosov element of the mapping class group has a power that is given by this construction [Pen88]. This is a difficult conjecture to disprove. The idea of Penner's proof of the theorem above is that one can explicitly find the train track associated to the square of any such element. The train track is obtained by smoothing out the subset of $A \cup B$ of S.

5.9 Nielsen-Thurston's Classification Theorem

Thurston's Construction has made it very see to show that $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$ is pseudo-Anosov. As T^2 can embedded in $\mathbb{R}^2$, we can prove that A is anosov as long as $\rho(A)$ is hyperbolic. To do this, however, we require Nielsen-Thurston's Classification Theorem, which helps us classify elements in $SL(2, \mathbb{Z})$ into elliptic, parabolic or hyperbolic functions.

In this subsection, we will introduce the Nielsen-Thurston Classification Theorem, which can be used to prove that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is indeed pseudo-Anosov.

The theorem is as follows:

Theorem 5.5. *Nielsen-Thurston Classification Let $g, n \geq 0$. Each mapping class $f \in$ Mod($S_{g,n}$) is periodic, reducible or pseudo-Anosov. Further, pseudo-Anosov mapping classes are neither periodic nor reducible.*

The proof for the general statement is quite involved. So, I shall focus on $\Sigma_{1,1}$. There are 2 approaches which we could use to classify the mapping classes of $\text{Mod}(\Sigma_{1,1})$ into one of the trichotomy: using Hyperbolic geometry or linear algebra. We will adopt the linear algebra approach as we do not have to build the theory of hyperbolic geometry. For readers interested in the hyberbolic geometry approach to classification can refer to [Mar12].

Just using the isomorphism $\text{Mod}(\Sigma_{1,1}) \cong SL(2, \mathbb{Z})$, we can give an algebraic approach to the classification for $\text{Mod}(\Sigma_{1,1})$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $f \in \text{Mod}(\Sigma_{1,1})$ denote the corresponding mapping class. The characteristic polynomial for A is $x^2 - tr(A)x + 1$. As the determinant of a matrix is equal to the product of it's eigenvalues and $\det(A) = 1$ as $A \in SL(2, \mathbb{Z})$, it follows that the eigenvalues of A are inverses of each other- call them λ_1 and $\lambda_2 = \frac{1}{\lambda_1}$.

Note that to find the eigenvalues λ_1 and λ_2, we need to solve the following equation:

$$det(\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}) = 0.$$

This leads to the characteristic polynomial:

$$\begin{aligned} &\lambda^2 - (a+d)\lambda + (ad-bc) = 0, \text{ which is equivalent to ,} \\ &\lambda^2 - tr(A)\lambda + det(A) = 0, \\ &\lambda^2 - tr(A)\lambda + 1 = 0. \end{aligned}$$

Making λ the subject of the equation results in:

$$\lambda = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4}}{2}.$$

The number of roots to the equation above classifies $(2, \mathbb{Z})$ into 3 disjoint partitions. Note that the number of roots correspond to whether or not the discriminant is more than, less than, or equal to 0.
Thus, there are 3 cases to consider: Then, there are 3 cases in total:

A. $tr(A)^2 - 4 > 0$,

B. $tr(A)^2 - 4 = 0$,

C. $tr(A)^2 - 4 < 0$.

These lead to the case:

A. $|\text{tr(A)}| > 0$,

B. $|\text{tr(A)}| = 0$,

C. $|\text{tr(A)}| < 0$.

The 3 cases are equivalent to the cases:
In case A, A has rational eigenvector, and from this it follows that f is reducible. In the 3 case, we see that A has 2 real eigenvalues, and so f is anosov.

In case B, λ and $\frac{1}{\lambda}$ are complex, $\lambda = \frac{1}{\lambda} = \pm 1$, and λ and $\frac{1}{\lambda}$ are distinct reals.

In case C, it follows from the Cayley-Hamiliton Theorem that A, hence f, has finite order.

I will elaborate on the first case as that is the emphasis in this project. Given, 2 eigenvalues E_{λ_1} and E_{λ_2}, we can have 2 linearly independent eigenspaces E_{λ_1} and E_{λ_2}. Thus, every vector v, v can be written as a linear combination of eigenvectors v_{λ_1} and v_{λ_2}, which span the corresponding eigenspaces E_{λ_1} and E_{λ_2}. Then, we can see that, the action of A on $\mathbb{R}^2$ results in:

$$\begin{aligned} A(v) &= A(\mu_1(v_{\lambda_1}) + \mu_2(v_{\lambda_2})) \\ &= A(\mu_1(v_{\lambda_1})) + A(\mu_2(v_{\lambda_2})) \\ &= \mu_1 A((v_{\lambda_1})) + \mu_2 A((v_{\lambda_2})) \\ &= \mu_1\lambda_1(v_{\lambda_1}) + \mu_2\lambda_2(v_{\lambda_2}). \end{aligned}$$

Since we have established that in case A that $\lambda_2 = \frac{1}{\lambda_1}$, the above equation becomes:

$$\mu_1\lambda_1(v_{\lambda_1}) + \mu_2\lambda_2(v_{\lambda_2}) = \mu_1\lambda_1(v_{\lambda_1}) + \mu_2\frac{1}{\lambda_1}(v_{\lambda_2}).$$

Thus, we can see that whenever $tr(A) > 2$, it follows that one of the eigenspaces is stretched by a factor of λ and the other is contracted by a factor of λ. This data, which gives a bundle of information about $f \in \text{Mod}(T^2)$ and is called the Anosov package, matches with the story from the earlier sections.

So, as $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, the $tr(A) = 3 > 2$. Thus, A is pseudo-Anosov.

5.10 Construction of new pseudo-Anosov functions acting on $\Sigma_{1,1}$

So, by Thurston's construction, Penner's Construction and the Nielsen-Thurston Classification theorem, we see that we can build new pseudo-Anosov function g acting on $\Sigma_{1,1}$ by taking the composition:

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{m_1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n_2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{m_2} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n_k} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{m_j}$$

such that $tr(g) > 2$, where n_i and m_j are integers.

6 Construction of Finite type Covering Spaces

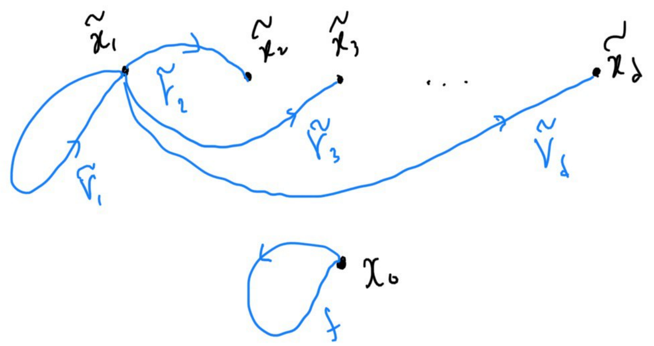
6.1 Monodromy Groups of Finite Covering

We will now study the monodromy representation, which allows us to encode information on how generators in $\pi(\Sigma_{1,1})$ act on the fibres of the base point of $\Sigma_{1,1}$ (the single one cell x) $p^{-1}(x)$, which allows us to construct finite type covering spaces. Note that this subsection is based on [Mir95]. So, readers interested can view this book for more details.

We will first explain the general case then talk about $\Sigma_{1,1}$. So, we shall use the notation X and x to denote the base space and it's fixed point. $\widetilde{X}$ and $\widetilde{x_0}$ will denote the covering space and its fixed point.

Let $F : \widetilde{X} \to X$ denote a connected covering space of finite degree d, such that all points in X have precisely d preimages. F corresponds to a subgroup $H \subset \pi_1(X, x_0)$ such that the degree d is the index of the subgroup H.

Consider the fibre $F^{-1}(x_0)$ over x_0. Let the d elements in this fiber be $\{\widetilde{x_1}, \widetilde{x_2}, ..., \widetilde{x_d}\}$. Every loop γ in X starting at x_0 can be lifted to d paths $\widetilde{\gamma_1}, \widetilde{\gamma_2}, ..., \widetilde{\gamma_d}$, where $\widetilde{\gamma_i}$ is the unique lift of γ which starts at $\widetilde{x_i}$. In other words, $\widetilde{\gamma}_i(0) = \widetilde{x_i}\ \forall i$.



Now, let's consider the endpoints of $\widetilde{\gamma}_i(1)$; these points also lie over x_0, and also form the entire preimage set $F^{-1}(x_0)$. Thus, each is an $\widetilde{x_j}$ for some j; we denote $\widetilde{\gamma}_i(1)$ by $\widetilde{x_{\sigma(i)}}$.

We can see that this function σ is a permutation of the indices $\{1, 2, ...d\}$, and it is only dependent on the homotopy class of the loop γ from the homotopy path lifting theorem in Algebraic Topology [Hat01]. Therefore, we have the group homomorphism

$$\begin{aligned} \rho : \pi_1(X, x_0) &\to S_d \\ [f] &\mapsto \sigma_f \end{aligned}$$

where S_d denotes the symmetric group of all permutations on d indices.

Definition 6.1 (Monodromy Representation of a Covering Space)**.** *The monodromy representation of a covering map $F : \widetilde{X} \to X$ of finite degree d is the group homomorphism $\rho : \pi_1(X, x_0) \to S_d$ defined above.*

Theorem 6.1. *Let $\rho : \pi_1(X, x_0) \to S_d$ be the monodromy representation of a covering map $F : \widetilde{X} \to X$ of finite degree d, with $\widetilde{X}$ connected. Then, the image of ρ is a transitive subgroup of S_d.*

Proof. With the notations we used above, fix 2 indices i and j, and consider 2 points $\tilde{x}_i$ and $\tilde{x}_j$ in the fibre of F over x_0. Since $\widetilde{X}$ is connected, a path $\tilde{\gamma}$ can be found on $\widetilde{X}$ starting at $\tilde{x}_i$ and ending at $\tilde{x}_j$. Let $\gamma = F \circ \tilde{\gamma}$ be the image of $\tilde{\gamma}$ in X. We can also see that γ is a loop in X based at x_0, since $\tilde{x}_i$ and $\tilde{x}_j$ map to x_0 under F. Then, by construction, $\rho([\gamma])$ is a permutation that sends i to j.

∎

6.2 Construction of Covering Spaces using Monodromy Representation

In the process above, we are sending a covering space and the covering map to its monodromy representative. We will see in the following subsection that we can also essentially give an inverse to this mapping: With a monodromy representation, we can build a covering space and its covering map. Transitivity is an important condition as transiitivity of the subgroup of S_n ensures the connectedness of our covering graph.

Now, we will construct covers using monodromy representation.

Suppose we have a connected real manifold X with base point x_0 and a group homomorphism $\rho : \pi_1(X, x_0) \to S_d$, from the fundamental group of X to a symmetric group S_d, with a transitive image. Fix an index: in this case: let it be 1. Let $H \subset \pi_1(X, x_0)$ be the subgroup consisting of the homotopy classes $[\gamma]$ such that $\rho([\gamma])$ fixes the index 1:

$$H = \{[\gamma] \in \pi_1(X, x_0) : \rho([\gamma]) = 1\}.$$

Then, H has index d in $\pi_1(X, x_0)$, and it induces a connected covering space $F_p : X_p \to X$. This can be verified by basic facts in [Hat01]. Further more, this covering has the property that its monodromy representation is exactly the given homomorphism ρ.

Hence, for a real connected manifold X, we have a 1-1 correspondence between the isomorphism classes of connected coverings $F : U \to V$ of degree d and group homomorphims $\rho : \pi_1(V, q) \to S_d$ with transitive image (up to conjugacy in S_d). The reason for conjugacy in S_d is easy to see: this simply reflects a relabelling of the points in the fibre of the covering over the base point.

In this section, I have shown how to build covering graphs from a given tree based on monodromy representation. The fundamental group of the covering graph is described by the set below:

$$H = \{[\gamma] \in \pi_1(X, x_0) : \rho([\gamma]) = 1\}.$$

In the next section, we will see that this H is a free group for the base surface $\Sigma_{1,1}$. To aid further calculations, we need to find the generators of H. This will be done by the theorems from Combinatorial Group Theory. However, before we do that, I will showcase an example of the monodromy from our software.

6.3 Worked out example of Monodromy Representation

In this section, I will elaborate on how the algorithms constructs 4-sheeted covers for the base space $\Sigma_{1,1}$ step by step. This will showcase how the algorithm works in depth.

Given $n = 4$, the algorithm generates S_4, the permutation group of size 4. The size of $S_4 = 24$. From the 24 covers, we understand that there are $\binom{24}{2} = 276$ possible pairing of permutations that generators a and b can be mapped to.

Let the vertices $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ and $(0, 0, 0, 1)$ be denoted as $1, 2, 3$ and 4. So, an edge from $(1, 0, 0, 0)$ to $(0, 0, 1, 0)$ will be denoted as $(1, 3)$.

We will focus on the following 3 representations and build the corresponding covers.

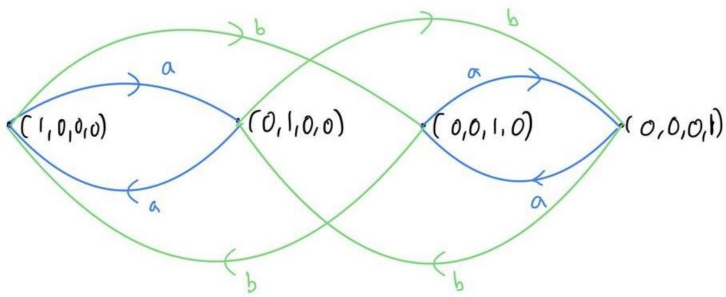
The ρ function of the **first cover** is as follows:

- $a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
- $b \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

We can see that the action of the matrix associated to a and b sends:

- $1 \mapsto 2$ and 3 respectively,
- $2 \mapsto 1$ and 4 respectively,
- $3 \mapsto 4$ and 1 respectively,
- $4 \mapsto 3$ and 2 respectively.

The resulting cover is:



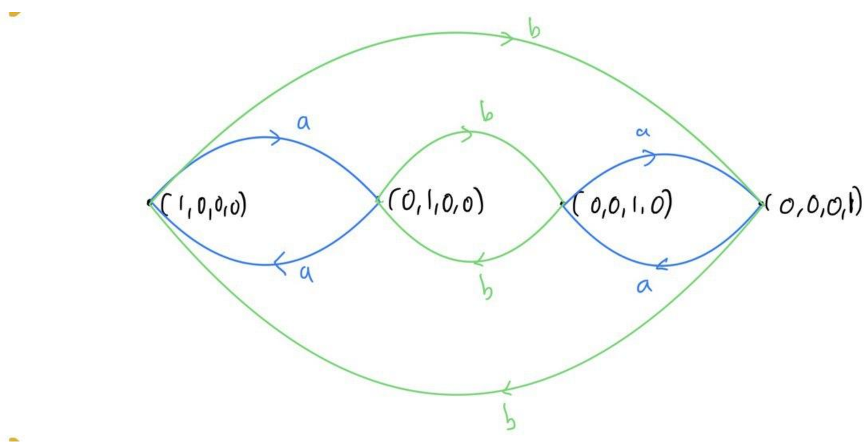
The ρ function of the **second cover** is as follows:

- $a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
- $b \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

We can see that the action of the matrix associated to a and b sends:

- $1 \mapsto 2$ and 4 respectively,
- $2 \mapsto 1$ and 3 respectively,
- $3 \mapsto 4$ and 2 respectively,
- $4 \mapsto 3$ and 1 respectively.

The resulting cover is:



The ρ function of the **third cover** is as follows:

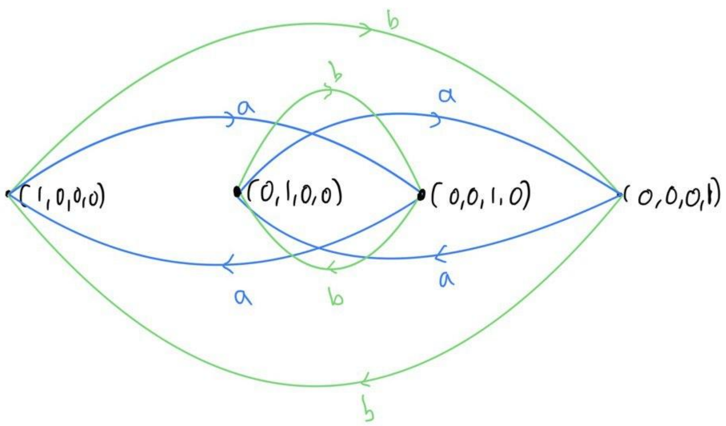
- $a \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

- $b \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

We can see that the action of the matrix associated to a and b sends:

- $1 \mapsto 3$ and 4 respectively,
- $2 \mapsto 4$ and 3 respectively,
- $3 \mapsto 1$ and 2 respectively,
- $4 \mapsto 2$ and 1 respectively.

The resulting cover is:



6.4 Finding the Fundamental Group of the Covering Graph

In the above section, we constructed the finite type covering graphs. We now need to find the generators of the fundamental group of the covering graphs. Finding the generators is an important step as the set of generators will be used to check if f can be lifted to their finite covers in the next section. Note that this subsection is based on [Sti93]. So, readers interested can view this book for more details.

Theorem 6.2. *Let X be a connected graph with vertex set X^0. Let $\widetilde{X}$ be the covering graph of X and $p : \widetilde{X} \to X$ be the covering map. Let $\widetilde{X}^0 = p^{-1}(X^0)$. Then, $\widetilde{X}$ is a graph with vertex set $\widetilde{X}^0$.*

Proof. First, note that $\widetilde{X}^0$ is a closed, discrete subset of $\widetilde{X}$. Let e be an edge in X. Note that each edge of $p^{-1}(e)$ is mapped homeomorphically onto e as covering spaces are locally homeomorphic. Also, each component of $p^{-1}(e)$ is open in $p^{-1}(e)$, by the local connectivity property. Thus, the condition (b) from the definition of graphs hold. Likewise, it is easy to verify condition (c); if $\overline{e}$ is homeomorphic to $[0, 1]$, then each edge of $p^{-1}(e)$ is mapped homeomorphically. If $\overline{e}$ is homeomorphic to S^1, then we apply what we know about covering spaces of a circle. Finally, condition (d) is a direct consequence of the following theorem:

Assume $(\widetilde{Y}, p)$ is a regular covering space of Y. If Y has the largest topology which makes all the maps f_λ continuous, then $\widetilde{Y}$ has the topology that makes all the maps f_{λ_i} continuous, where $\{f_{\lambda_i}\}$ denote the set of all possible liftings of f_λ.

, because there is a map $f_i : \mathbb{I} \to X$, corresponding to any edge e_i, such that $f(\mathbb{I}) = \overline{e_i}$) and f_i maps the open interval $(0, 1)$ homeomorphically onto e_i.

■

We need to calculate the fundamental group of the covering graph $\pi(\tilde{X}, \tilde{x_0})$. This will be done based on the Nielsen-Schreier Theorem and the Theory of Schreier Transversals.

Theorem 6.3. *The Fundamental group of any connected graph X is a free group.*

Proof. The theorem is obvious is X is a tree, because the fundamental group is trivial. For the case where X is not a tree, we shall prove the following more explicit theorem.

First note that a edge path in a graph has an rather obivous relation to paths in a topological space: An edge path $(e_1 e_2 ... e_n)$ in the graph X joining vertices v_0 and v_1 determines a unique equivalence class of paths in the topological space X joining the points v_0 and

v_1, as follows. For each oriented edge e_i, choose a map $f_i : \mathbb{I} \to \overline{e_i}$ such that $f_i|(0,1)$ is a homeomorphism of $(0,1)$ onto e_i whose inverse belongs to the preferred equivalence class determined by the orientation of e_i. Let α_i denote the equivalence class of the path f_i. Then, the product $\alpha_1\alpha_2...\alpha_n$ is uniquely determined by the edge path $(e_1e_2...e_n)$.

Let X be a connected graph, let v_0 be a vertex of X, and let T be the maximal tree in X containing v_0. Let $\{e_\lambda : \lambda \in \Delta\}$ denote the set of edges in X not contained in T. Choose a definite orientation for each of the edges e_λ; let a_λ and b_λ denote the initial and terminal vertices of e_λ (It may be the case where $a_\lambda = b_\lambda$). To each edge e_λ, we associate an element $\alpha_\lambda \in \pi(X, v_0)$ as follows. There is a unique reduced edge path A_λ in T from v_0 to a_λ and a unique reduced edge path B_λ in T from b_λ to v_0. Then, α_λ is the path class associated with the edge path $(A_\lambda, e_\lambda, B_\lambda)$. if $a_\lambda = v_0$, we omit A_λ; similarly if $b_\lambda = v_0$ we omit B_λ.

■

Now that we have seen that the fundamental group of the covering graph is in fact a free group, we know that we can express the fundamental group in terms of generators. We shall now see how the construction of the generators is done.

Theorem 6.4. *The fundamental group $\pi(X, v_0)$ is a free group on the set of generators $\{\alpha_\lambda : \lambda \in \Delta\}$.*

Proof. First, we prove this theorem for the case where the index set Δ contains only 1 element; that is to say there is only 1 edge of X not contained in T. We denote this edge by e_1. Because X is not a tree, there exist closed edge paths in X, and it is clear that any such closed path must involve the edge e_1. Give the edge e_1 a definite orientation. Then, there must exist reduced closed edge paths in X starting with e_1, that is to say edge paths of the form $(e_1e_2...e_n)$. By choosing the shortest among all such closed edge paths, we obtain a simple closed edge path, meaning an edge path with no repeated edges or vertices. Denote this simple closed edge path by $(e_1e_2...e_m)$. Let

$$C = \cup_{i=1}^{m}\overline{e_i}.$$

Then, C is a subgraph of X homeomorphic to a circle.

Consider the complementary set $X - C$; let $\{Y_i\}$ denote the set of components of $X - C$. Each $\overline{Y_i}$ is a subgraph of T; hence, it is a tree. An easy argument shows that $\overline{Y_i}$ has exactly one vertex in common with C. Each of the trees $\overline{Y_i}$ can be contracted to this vertex. This shows the deformation retract of X; hence, the inclusion map $C \to X$ induces an isomorphism of fundamental groups. This shows that $\pi(X)$ is an infinite cyclic group.

Choose a point $x_\lambda \in e_\lambda$ for each $\lambda \in \Delta$. The set $\{x_\lambda : \lambda \in \Delta\}$ is closed and discrete because X has weak topology. Let U denote the complement of $\{\alpha_\lambda : \lambda \in \Delta\}$ in X. Then ,T is the deformation retract of U; hence U is contractible. For any index λ, let

$$V_\lambda = U \cup \{x_\lambda\}$$

Then, $U \subset V_\lambda$ for all λ, and, if $\lambda \neq \mu$,

$$V_\lambda \cap V_\mu = U.$$

Clearly, V_λ has $T \cup e_\lambda$ as a deformation retract; this, the fundamental group $\pi(V_\lambda, v_0)$ is infinite cyclic and generated by α_λ.

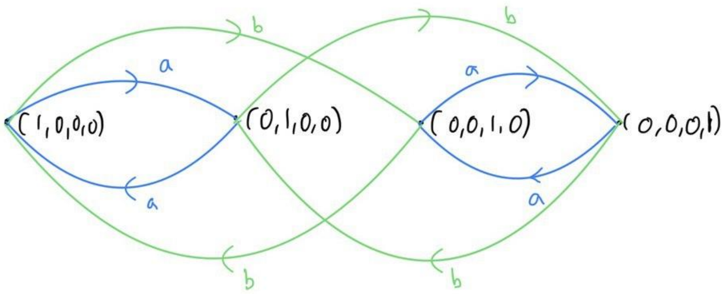
Then, we can conclude that $\pi(X, v_0)$ is the free product of the groups $\pi(V_\lambda, v_0)$.

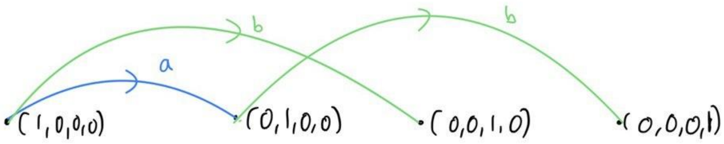
■

6.5 Worked out example of Nielsen-Schreier Theorem in the software

In the software, this is how we determine the generators of the fundamental group of the covering graph:

A. Find the spanning Tree of the Covering Graph. For example, the spanning tree of the first example of the 4-sheeted cover is:





The edges that make up the spanning tree are: $[a, b, ab]$.

B. Identify the edges not in the spanning tree

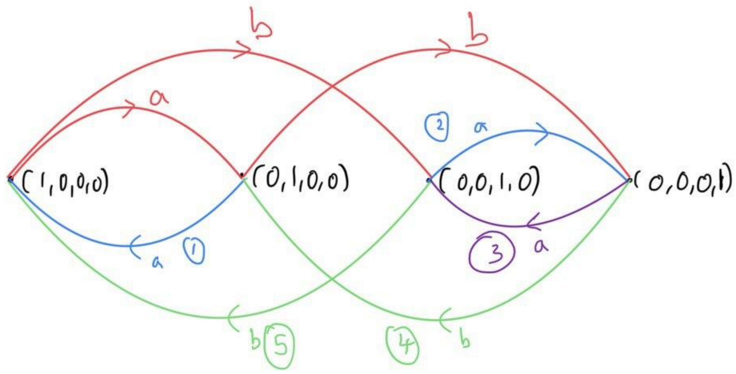
C. Generate loops using the spanning tree and the edges not in the spanning tree

(a) Choose a definite orientation for each of the edges e_λ

(b) let a_λ and b_λ denote the initial and terminal vertices of e_λ (It may be the case where $a_\lambda = b_\lambda$)

(c) There is a unique reduced edge path A_λ in T from v_0 to a_λ and a unique reduced edge path B_λ in T from b_λ to v_0

(d) α_λ is the path class associated with the edge path $(A_\lambda, e_\lambda, B_\lambda)$

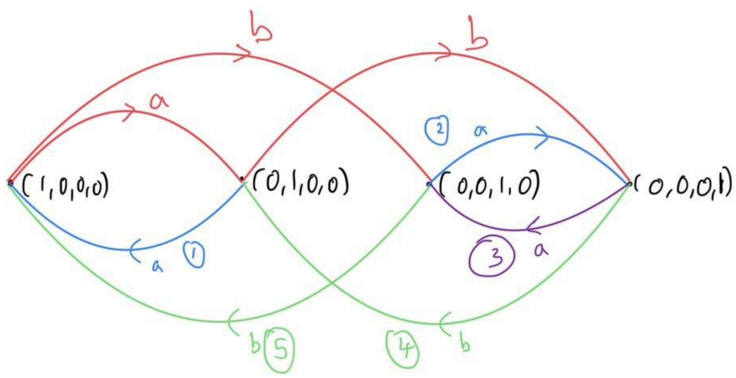


A. The edges in Red denote the edges of the spanning tree

B. The initial and final vertex of edge 3 are $[0, 0, 0, 1]$ and $[0, 0, 1, 0]$ respectively.

C. Suppose now we want to build the generator that corresponds to edge 3. There is a unique path in the spanning tree to $[0, 0, 0, 1]$ and $[0, 0, 1, 0]$ respectively: ab and b

D. Thus, the corresponding loop that generates $\pi(\widetilde{X}, [1, 0, 0, 0])$ is $abab^{-1}$.



- We can see that there are a total of 5 edges not contained in the spanning tree
- Each of them correspond to 1 generator of $\pi(\widetilde{X}, [1, 0, 0, 0])$ is $abab^{-1}$
- So, for this cover, the generators of $\pi(\widetilde{X}, [1, 0, 0, 0])$ are:

A. aa
B. $bab^{-1}a^{-1}$

C. $abab^{-1}$

D. $abba^{-1}$

E. bb

So, we can see that for the **first cover**, the generators of the fundamental group of the covering graph are:

$$1)abab^{-1}$$
$$2)abba^{-1}$$
$$3)aa$$
$$4)bab^{-1}a^{-1}$$
$$5)bb.$$

For the **second cover**, we can see that the generators of the fundamental group of the covering graph are:

$$1)aa$$
$$2)abab^{-1}$$
$$3)abba^{-1}$$
$$4)bab^{-1}a^{-1}$$
$$5)bb.$$

For the **third cover**, we can see that the generators of the fundamental group of the covering graph are:

$$1)abab^{-1}$$
$$2)abba^{-1}$$
$$3)aa$$
$$4)bab^{-1}a^{-1}$$
$$5)bb.$$

7 Homological Representations and Fox Derivatives

Thus far, we seen how to find the mapping class group of a surface, $\Sigma_{g,n}$ and construct pseudo-Anosov maps for the surface. We have also seen how to construct finite type covers, $\widetilde{\Sigma_{1,1}}$, for $\Sigma_{1,1}$. Next, for finite type covers constructed, we need to figure out a way construct the lift of the function. The ideas in this chapter are mostly motivated and edited from the contents in [Zie13].

To do this, I employed a slight variation of the Fox Derivatives. Before explaining that, I will first talk about homological representations.

7.1 Introduction to Homological Representations

The virtual homological spectral radius of a pseudo-Anosov map f is calculated from the homological representation of the lift of f from the base space $\Sigma_{1,1}$ to the covering graph, $\widetilde{\Sigma_{1,1}}$.

Homological representations are finite dimensional representations of $\text{Mod}(\Sigma_{g,n})$, which are associated to finite covers $\pi : \widetilde{\Sigma_{g,n}} \to \Sigma_{g,n}$. The standard homological representation is:

$$\text{Mod}(\Sigma_{g,n}) \to GL(H_1(\Sigma_{g,n}, \mathbb{Q}))$$

given by the induced action on the first homology.

More generally, fix a base point $*$ in $\Sigma_{g,n}$. So, the mapping class group of $\Sigma_{g,n}$ is $\text{Mod}(\Sigma_{g,n}, *)$. This group, $\text{Mod}(\Sigma_{g,n}, *)$, acts on $\pi_1(\Sigma_{g,n}, *)$ by automorphisms. Let K be a finite index subgroup of $\pi_1(\Sigma_{g,n}, *)$, and let $\pi : \widetilde{\Sigma_{g,n}} \to \Sigma_{g,n}$ be the associated finite cover. Let $G_k = \{f \in Mod\Sigma_{g,n}S, *) : f(K) = K\}$. The group G_k is a finite index subgroup of $\text{Mod}(\Sigma_{g,n}, *)$. We have the natural map $G_k \to GL(H_1(\widetilde{\Sigma_{g,n}}, \mathbb{Q}))$ Topologically, every element $f \in G_k$ can be lifted to $f' : \widetilde{\Sigma_{g,n}} \to \widetilde{\Sigma_{g,n}}$. The lift induces a map $f'_* : GL(H_1(\widetilde{\Sigma_{g,n}}, \mathbb{Q})) \to GL(H_1(\widetilde{\Sigma_{g,n}}, \mathbb{Q}))$. The transformation f'_* is $p_k(f')$. The representations p_k are called homological representations [Had20].

Now, that we have the collection of covers which lift to the covering graphs, we can start constructing the homological representations of the induced homology map of $\widetilde{f}$, $\widetilde{f}_*$. The method of construction is based on the theory of Fox derivations in the group ring of a free group. There is a geometric background of Fox Calculus with which we intend to start with.

7.2 Regular Coverings and Homotopy chains

Let $\rho : \widetilde{X} \to X$ be a regular covering of a connected 2-complex, where X is a finite CW-complex with one 0-cell P. Then, a presentation

$$B = \pi_1(X, P) =< s_1, s_2, ..., s_n | R_1, R_2, ..., R_n >$$

of the fundamental group of X is attained by assigning a generator s_i to each (oriented) 1-cell (which is represented by s_i too), and a defining relation to (the boundary of) each 2-cell e_j of X. Choose a base point $\widetilde{P} \subset \widetilde{X}$ over P, $p_*(\pi_1(\widetilde{X}, \widetilde{P})) = N \lhd B$, and let $D \cong B/N$ represent the group of covering transformations.

Let $\phi : B \to D$, $w \mapsto w^\phi$ be the standard homomorphism.

$$\begin{aligned} \phi : \pi_1(X, P) &\to \frac{\pi_1(X, P)}{\pi_1(\widetilde{X}, \widetilde{P}))}, \\ [f] &\mapsto [f] \cdot \pi_1(\widetilde{X}, \widetilde{P})) \end{aligned}$$

Note that $(w_1 w_2)^\phi = (w_1)^\phi (w_2)^\phi$. That is to say:

$$([f_1][f_2]) \cdot \pi_1(\widetilde{X}, \widetilde{P})) = ([f_1] \cdot \pi_1(\widetilde{X}, \widetilde{P})))([f_2] \cdot \pi_1(\widetilde{X}, \widetilde{P}))$$

The linear extension to the group ring is given by $\phi : \mathbb{Z}B \to \mathbb{Z}D$.

$$\begin{aligned} \phi : \mathbb{Z}[\pi_1(X, P)] &\to \mathbb{Z}[\frac{\pi_1(X, P)}{\pi_1(\widetilde{X}, \widetilde{P}))}], \\ m[f] &\mapsto m([f] \cdot \pi_1(\widetilde{X}, \widetilde{P}))) \end{aligned}$$

Our goal is to present $H_1(\widetilde{X}, \widetilde{X^0})$ as a $\mathbb{Z}D$-module. (We follow the common convention by writing D-module instead of $\mathbb{Z}D$-module. $\widetilde{X^0}$ is the 0-skeleton of $\widetilde{X}$.) This means that we want elements in $\frac{\pi_1(X,P)}{\pi_1(\widetilde{X},\widetilde{P}))}$, which represent covering transformations on the cover of X to act on the group $H_1(\widetilde{X}, \widetilde{X^0})$ by the operation of scalar multiplication. We will now see how that can be done.

The (oriented) edges s_i lift to edges $\widetilde{s_i}$ with initial point $\widetilde{P}$. By w, we denote both a closed path in the 1-skeleton X^1 of X, and the elements it represents in the free group $F = \pi_1(X^1, P) =< s_1, s_2, ..., s_n| - >$. There exists a unique lift $\widetilde{w}$ of w starting at $\widetilde{P}$. $\widetilde{w}$ is a special element of the relative cycles $Z_1(\widetilde{X}, \widetilde{X^0})$ which are called homotopy 1-chains.

Every 1-chain can be written in the form $\sum_{j=1}^{n} \epsilon_j \widetilde{s_j}$, where $\epsilon_j \in \mathbb{Z}D$. There is a rule

$$\widetilde{w_1 w_2} = \widetilde{w_1} + w_1^{\phi} \cdot \widetilde{w_2}.$$

To understand it, first lift w_1 to $\widetilde{w_1}$. Its endpoint is $w_1^{\phi} \cdot \widetilde{P}$. The covering transformation w_1^{ϕ} maps $\widetilde{w_2}$ onto a chain $w_1^{\phi} \cdot \widetilde{P}$.

If $\widetilde{w_k} = \sum_{j=1}^{n} \epsilon_j \widetilde{s_j}$, $k = 1, 2, ...$, then $\widetilde{w_1 w_2} = \sum_{j=1}^{n} \epsilon_j \widetilde{s_j}$ with

$$\epsilon_j = \epsilon_{1j} + w_1^{\phi} \dot{\epsilon}_{2j}, 1 \leq j \leq n.$$

This defines mappings:

$$(\frac{\partial}{\partial(s_j)})^{\phi} : B = \pi_1(X, P) \to \mathbb{Z}D$$
$$w \mapsto \epsilon_j, \text{ with } \widetilde{w} = \sum_{j=1}^{n} \epsilon_j \widetilde{s_j},$$

satisfying the rule

$$(\frac{\partial}{\partial(s_j)}(w_1 w_2))^{\phi} = (\frac{\partial}{\partial(s_j)}(w_1))^{\phi} + w_1^{\phi}(\frac{\partial}{\partial(s_j)}(w_2))^{\phi}.$$

There is a linear extension to the group ring $\mathbb{Z}B$:

$$(\tfrac{\partial}{\partial s_j}(\eta + \epsilon)))^{\phi} = (\tfrac{\partial}{\partial s_j}(\eta))^{\phi} + (\tfrac{\partial}{\partial s_j}(\epsilon))^{\phi}$$

for $\eta, \epsilon \in \mathbb{Z}B$.

From the definition, it follows immediately that

$$(\tfrac{\partial}{\partial s_j}(s_k))^{\phi} = \delta_{jk}, \ \widetilde{w} = \sum (\tfrac{\partial w}{\partial s_j})^{\phi} (\widetilde{sj}),$$

where δ_{jk} is the Kronecker delta.
We may now use this terminology to present $H_1(\widetilde{X}, \widetilde{X^0})$ as a D-module. The 1-chains $\widetilde{s_i}, 1 \leq i \leq n$, are generators and the lifts $\widetilde{R_j}$ of the boundaries $R_j = e_j$ of the 2-cells are defining the relations.

Theorem 7.1. *$H_1(\widetilde{X}, \widetilde{X^0}) =< \widetilde{s_1}, \widetilde{s_2}, ..., \widetilde{s_n} | \widetilde{R_1}, \widetilde{R_2}, ..., \widetilde{R_m} >$, $0 = \widetilde{R_j}$ is a presentation of $H_1(\widetilde{X}, \widetilde{X^0})$ as a D-module.*

Now, we introduce the Fox Differential Calculus. Here, we will descirbe the purely algebraic approach to the mapping $(\frac{\partial}{\partial s_j})^{\phi}$. Let B be a group and $\mathbb{Z}B$ its group ring with integer coefficients. We will skip the proofs of the theorem in this subsection. Anyone interested in the proofs can refer to [Zie13].

7.3 Fox Differential Calculus

This section serves more as a check to manually see by hand if the results we derive is correct.

Definition 7.1 (Augmentation Homomorphism)**.** *There is a homomorphism $\epsilon : \mathbb{Z}B \to \mathbb{Z}$, $\tau = \sum n_i g_i \mapsto \sum n_i = \tau^{\epsilon}$. We call ϵ the augmentation homomorphism, and its kernal $\mathbb{I}B = \epsilon^{-1}(0)$ the augmentation ideal.*

Definition 7.2. *(Derivations) A mapping $\Delta : \mathbb{Z}B \to \mathbb{Z}B$ is called a derivation (of $\mathbb{Z}B$) if:*

- $\Delta(\epsilon + \eta) = \Delta(\epsilon) + \Delta(\eta)$ *and*
- $\Delta(\epsilon\dot{\eta}) = \Delta(\epsilon)\dot{\eta}^{\epsilon} + \epsilon\dot{\Delta}(\eta)$

for $\epsilon, \eta \in \mathbb{Z}B$.

Theorem 7.2. *The derivations of $\mathbb{Z}B$ form a (right) B-module under the operations defined by*

$$(\Delta_1 + \Delta_2)(\tau) = \Delta_1(\tau) + \Delta_2(\tau), \text{ and}$$
$$(\Delta\gamma)(\tau) = \Delta(\gamma)\dot{\gamma}.$$

Theorem 7.3. *Let Δ be a derivation. Then:*

- $\Delta(m) = 0, \forall m \in \mathbb{Z}$,
- $\Delta(g^{-1}) = -g^{-1}\dot{\Delta}(g)$,
- $\Delta(g^n) = (1 + g + ... + g^{n-1})\dot{\Delta}(g)$,
- $\Delta(g^{-n}) = -(g^{-1} + g^{-2} + ... + g^{-n})\dot{\Delta}(g)$, *for* $n \geq 1$.

Theorem 7.4. *Let* $F = < s_i | i \in J >$ *be a free group. There exists a uniquely determined derivation* $\Delta : \mathbb{Z}F \to \mathbb{Z}F$ *with* $\Delta(S_i) = w_i$ *for arbitrary elements* $w_i \in \mathbb{Z}F$.

Definition 7.3 (Partial Derivatives)**.** *The derivations*

$$\frac{\partial}{\partial S_i} : \mathbb{Z}F \to \mathbb{Z}F,$$

of the group ring of a free group $F =< \{s_i | i \in J\} | - >$ *are called partial derivations.*

The partial derivations form a basis of the modules of derivations:

Theorem 7.5.

- $\Delta = \sum_{i \in J} \frac{\partial}{\partial S_i}\dot{\Delta}(S_i)$ *for every derivation* $\Delta : \mathbb{Z}F \to \mathbb{Z}F$.
- $\Delta = \sum_{i \in J} \frac{\partial}{\partial S_i}\dot{\tau}_i = 0 \leftrightarrow \tau_i = 0, \forall i \in J$.
- $\Delta_\epsilon(\tau) = \tau - \tau^\epsilon = \sum_{i \in J} \frac{\partial \tau}{\partial S_i}\dot{(}S_i - 1)$
- $\tau - \tau^\epsilon = \sum_{i \in J} v_i(S_i - 1) \leftrightarrow v_i = \frac{\partial \tau}{\partial S_i}, i \in J$.

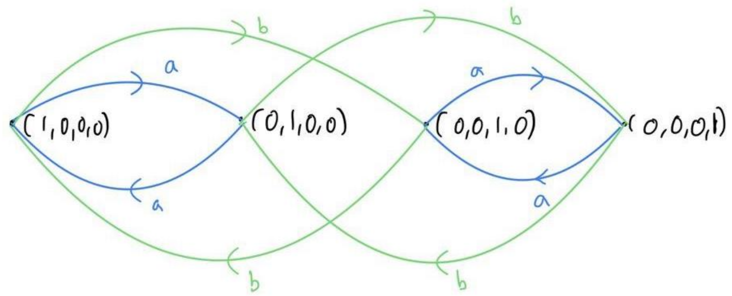
7.4 Application of Fox Calculus to get the Virtual Homological Spectral Radii

The most important takeaway from the subsections above, is that:

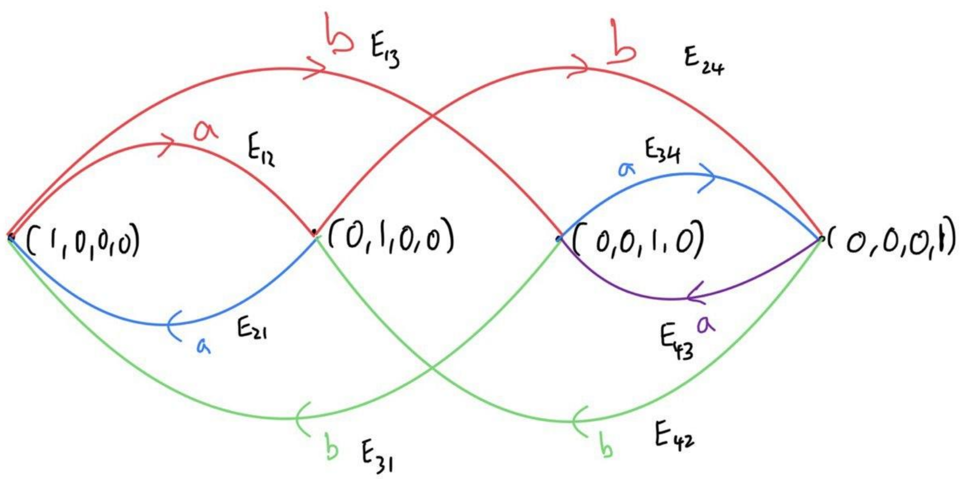
$$\widetilde{w_1 w_2} = \widetilde{w_1} + w_1^{\phi} \cdot \widetilde{w_2}.$$

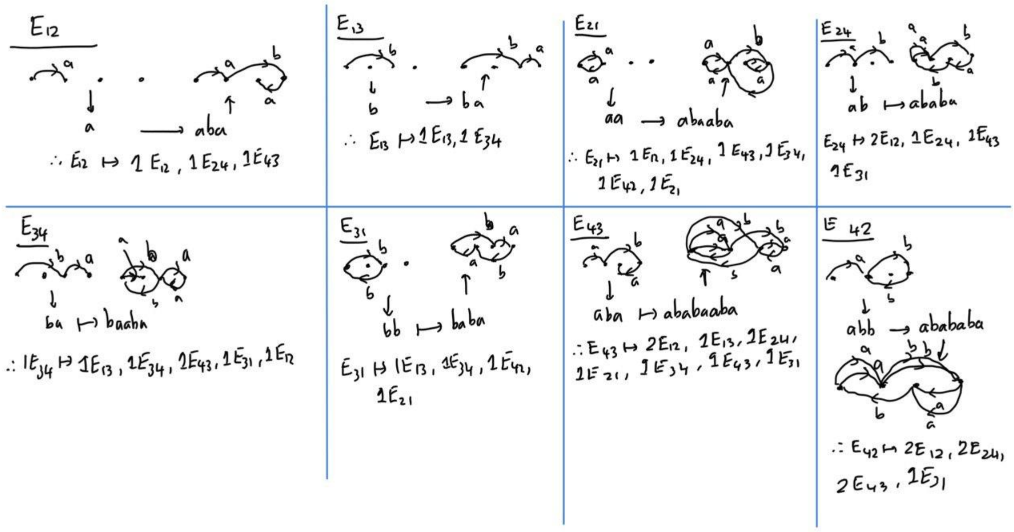
This system of storing information of how the homology map acts.

I will show how we construct the homological matrix for the following cover:



- In our software, the contruction is mechanical. Given an edge $e \in \widetilde{X}$:
  A. we take the path in the spanning tree to e and join them together. Let's call this concatenation γ.
  B. Using the covering map $p : \widetilde{X} \to X$, we project the path γ on X.
  C. Then, f acts on γ.
  D. We unwarp $f(\gamma)$ as a path in $\widetilde{X}$ starting at the basepoint $[1, 0, 0, 0]$.
  E. The linear combination of resulting edges in $\widetilde{X}$ forms the row in our homological matrix. The rows and columns in our matrix is indexed by the edges forming our covering graph.
  F. Lastly, we remove the edges and columns of the spanning tree, thus forming our homological matrix.





$E_{1,2}$**:** We can see that the edge $E_{1,2}$ in the covering graph is mapped down to the generator a in the base space first. As f acts on a, a is mapped to 1 copy of a, 1 copy of b and 1 copy of a, in order. We shall span this path above in the covering graph starting from our base point. a maps $[1,0,0,0]$ to $[0,1,0,0]$ at first, then b maps $[0,1,0,0]$ to $[0,0,0,1]$ and lastly a maps $[0,0,0,1]$ to $[0,0,1,0]$. Thus, we get the respective edges: $E_{1,2}$, $E_{2,4}$ and $E_{4,3}$. Thus,

$$E_{1,2} \mapsto E_{1,2}, E_{2,4} \text{ and } E_{4,3}$$

We carry out this process for each generator of the fundamental group of the covering space to see where $\widetilde{f}$ each edge to.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The row of the matrix are indexed accordingly from top to bottom with respect to the generators of the fundamental group of the covering graph: aa, $bab^{-1}a^{-1}$, $abab^{-1}$, $abba^{-1}$, bb.

The column of the matrix is indexed accordingly from left to right: $[E_{12}, E_{13}, E_{21}, E_{24}, E_{31}, E_{34}, E_{42}, E_{43}]$.

To get the homological representation, we have to remove the edges in the spanning tree of the covering graph from the column of the matrix constructed above. This gives us the homological representation:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

8 Results

In this section, I will elaborate on how the algorithms work and demonstrate step-by-step for the case of 4-sheeted covers for the base space $\Sigma_{1,1}$.

8.1 General Information

In the first step, the algorithm will prompt 2 information: an integer n, and the generators of the base space. As we are dealing with 4-sheeted covers of $\Sigma_{1,1}$, $n = 2$ and the generators are a, b.

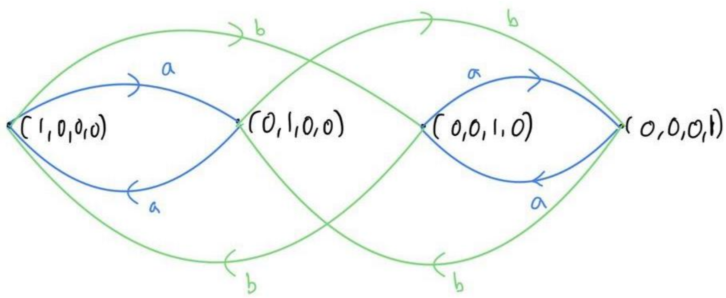
Given $n = 4$, the algorithm generates S_4, the permutation group of size 4. The size of $S_4 = 24$. Using monodromy representation [6.1], we will notice that there are a total of 276 covers generated. Out of these covers, we sieve through the list to get the covers which are connected. From the collection of connected covers, we then need to find the covers for which the pseudo-Anosov function f lifts. We will find that only 3, of the original 276 covers are connected and pass this lift criterion. I will use these 3 covers to elaborate on how the software works in detail. This is mainly reiterating the applications of the theories in the last few sections. However, this section was added to gain a more complete, bird's eye overview of the software.

8.2 First Cover

The ρ function of the first cover is as follows:

- $a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
- $b \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

The resulting covering graph is:



Then the software determines the generators of the fundamental group of the covering graph. So, for the first cover, the generators of $\pi(\widetilde{\Sigma_{1,1}})$ are:

A. aa

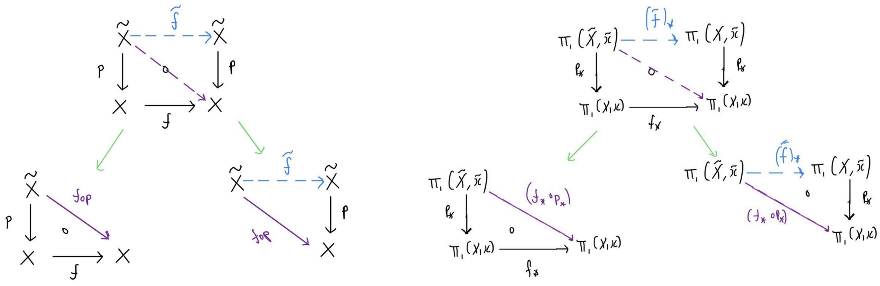
B. $bab^{-1}a^{-1}$

C. $abab^{-1}$

D. $abba^{-1}$

E. bb.

Now, the software will test if the pseudo-Anosov function f will lift for this cover, simply applying the standard lifting criterion theorem in Algebraic Topology [Hat01].



Thus, we can see that we need to check that $f_*p_*(\pi_1(\widetilde{\Sigma_{1,1}}) \subset p_*(\pi_1(\widetilde{\Sigma_{1,1}}))$. Thus, we have the following:

A. $f_*p_*(aa) = f_*(aa) = ababaabaa^{-1}b^{-1}$

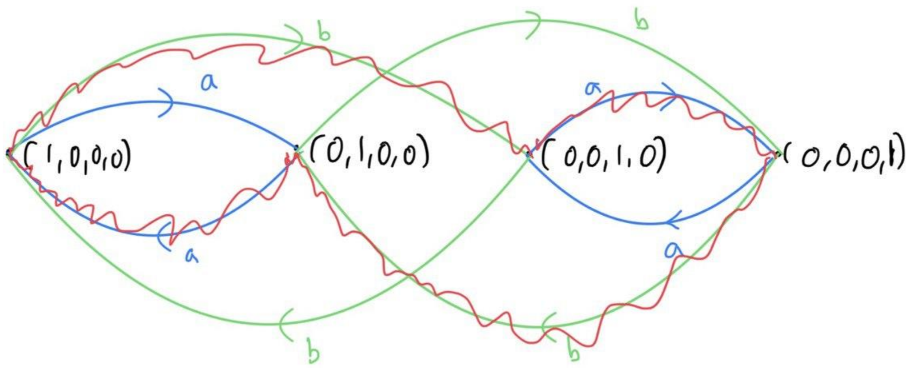
B. $f_*p_*(bab^{-1}a^{-1}) = f_*(bab^{-1}a^{-1}) = abababaa^{-1}b^{-1}a^{-1}$

C. $f_*p_*(abab^{-1}) = f_*(abab^{-1}) = abaaba$

D. $f_*p_*(abba^{-1}) = f_*(abba^{-1}) = baabaa^{-1}b^{-1}a^{-1}b^{-1}a^{-1}$

E. $f_*p_*(bb) = f_*(bb) = baba$.

However, it suffices to check that the words on the RHS maps back to the base point $[1, 0, 0, 0]$ as by the monodromy representation $\pi(\widetilde{\Sigma_{1,1}})$ is homomorphic to the subgroup $H = \{\sigma \in S_4 : \sigma(1) = 1\}$. To see how it works, let's focus on words E from the list above.



As the word in E is $baba$, the path starts at vertex $[1, 0, 0, 0]$, takes the edge b to vertex $[0, 0, 1, 0]$, then the edge a to vertex $[0, 0, 0, 1]$, edge b to vertex $[0, 1, 0, 0]$ and finally vertex a back to the initial vertex $[1, 0, 0, 0]$. So, we can see that path $baba$ is a loop around

$[1,0,0,0]$. We can see that this checks out for every element in the list above.

Now, we build the homological matrix for the lift. Using what we have learnt earlier, we can see that, the homological matrix can be constructed to obtain:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The row of the matrix are indexed accordingly from top to bottom with respect to the generators of the fundamental group of the covering graph: aa, $bab^{-1}a^{-1}$, $abab^{-1}$, $abba^{-1}$, bb.

The column of the matrix is indexed accordingly from left to right:
$[E_{12}, E_{13}, E_{21}, E_{24}, E_{31}, E_{34}, E_{42}, E_{43}]$.

From the matrix, we see that the eigenvalues are the following: 2.61, $-0.5 \pm 0.866i$ and 0.382. Thus, the spectral radius for this cover is: 2.61.

Applying the same idea on the other 2 4-sheeted covers of $\Sigma_{1,1}$, we can find the respective virtual homological spectral radius associated to each cover.

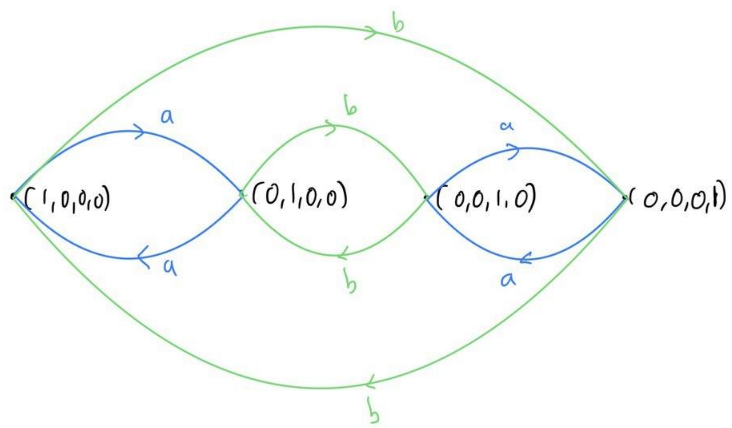
8.3 Second 4-sheeted Cover

The ρ function of the first cover is as follows:

- $a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

- $b \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

This results in the following cover:



Then the software determines the generators of the fundamental group of the covering graph. So, for the first cover, the generators of $\pi(\widetilde{\Sigma_{1,1}})$ are:

$$\begin{aligned} &1)aa \\ &2)abab^{-1} \\ &3)abba^{-1} \\ &4)bab^{-1}a^{-1} \\ &5)bb. \end{aligned}$$

Now, the software will test if the pseudo-Anosov function f will lift for this cover, simply applying the standard lifting criterion theorem in Algebraic Topology [Hat01]. Thus, we can see that we need to check that $f_* p_*(\pi_1(\widetilde{\Sigma_{1,1}}) \subset p_*(\pi_1(\widetilde{\Sigma_{1,1}}))$. Thus, we have the following:

$$\begin{aligned}
&1)abaaba \\
&2)ababaabaa^{-1}b^{-1} \\
&3)abababaa^{-1}b^{-1}a^{-1} \\
&4)baabaa^{-1}b^{-1}a^{-1}b^{-1}a^{-1} \\
&5)baba.
\end{aligned}$$

As mentioned in the earlier subsection, it suffices to check that the words on the RHS maps back to the base point $[1, 0, 0, 0]$ as by the monodromy representation $\pi(\widetilde{\Sigma_{1,1}})$ is homomorphic to the subgroup $H = \{\sigma \in S_4 : \sigma(1) = 1\}$.

The homological matrix of the lift, including the spanning tree edges, is:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}$$

The homological matrix of the lift is:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 2 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & -1 & 1 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix}$$

The row of the matrix are indexed accordingly from top to bottom with respect to the generators of the fundamental group of the covering graph: aa, $abab^{-1}$, $abba^{-1}$, $bab^{-1}a^{-1}$, bb.

The column of the matrix is indexed accordingly from left to right:
$[E_{12}, E_{14}, E_{21}, E_{23}, E_{34}, E_{31}, E_{43}, E_{41}]$.

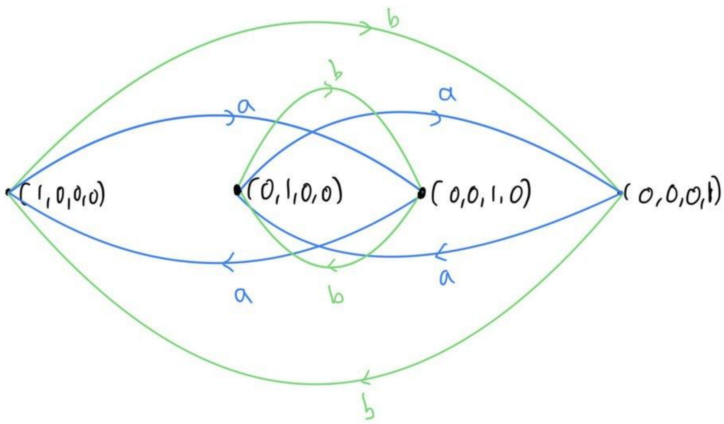
From the matrix, we see that the eigenvalues are the following: 2.61, $-0.5 \pm 0.866i$ and 0.382. Thus, the spectral radius for this cover is: 2.61.

8.4 Cover 3

The ρ function of the first cover is as follows:

- $a \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
- $b \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

The resulting cover is:



Then the software determines the generators of the fundamental group of the covering graph. So, for the first cover, the generators of $\pi(\widetilde{\Sigma_{1,1}})$ are:

$$\begin{aligned} &1)abab^{-1}\\ &2)abba^{-1}\\ &3)aa\\ &4)bab^{-1}a^{-1}\\ &5)bb. \end{aligned}$$

Now, the software will test if the pseudo-Anosov function f will lift for this cover, simply applying the standard lifting criterion theorem in Algebraic Topology [Hat01]. Thus, we can see that we need to check that $f_* p_*(\pi_1(\widetilde{\Sigma_{1,1}}) \subset p_*(\pi_1(\widetilde{\Sigma_{1,1}}))$. Thus, we have the following:

$$
\begin{aligned}
&1) ababaabaa^{-1}b^{-1} \\
&2) abababaa^{-1}b^{-1}a^{-1} \\
&3) abaaba \\
&4) baabaa^{-1}b^{-1}a^{-1}b^{-1}a^{-1} \\
&5) baba.
\end{aligned}
$$

As mentioned in the earlier subsection, it suffices to check that the words on the RHS maps back to the base point $[1, 0, 0, 0]$ as by the monodromy representation $\pi(\widetilde{\Sigma_{1,1}})$ is homomorphic to the subgroup $H = \{\sigma \in S_4 : \sigma(1) = 1\}$.

The homological matrix of the lift, including the spanning tree edges, is:

$$
\begin{pmatrix}
1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}
$$

The homological matrix of the lift is:

$$
\begin{pmatrix}
1 & 1 & 1 & 2 & 0 \\
0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & -1 & 0 & -1 & 1 \\
1 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

The row of the matrix are indexed accordingly from top to bottom with respect to the generators of the fundamental group of the covering graph: $abab^{-1}$, $abba^{-1}$, aa, $bab^{-1}a^{-1}$, bb.

The column of the matrix is indexed accordingly from left to right:
$[E_{13}, E_{14}, E_{24}, E_{23}, E_{31}, E_{32}, E_{42}, E_{41}]$.

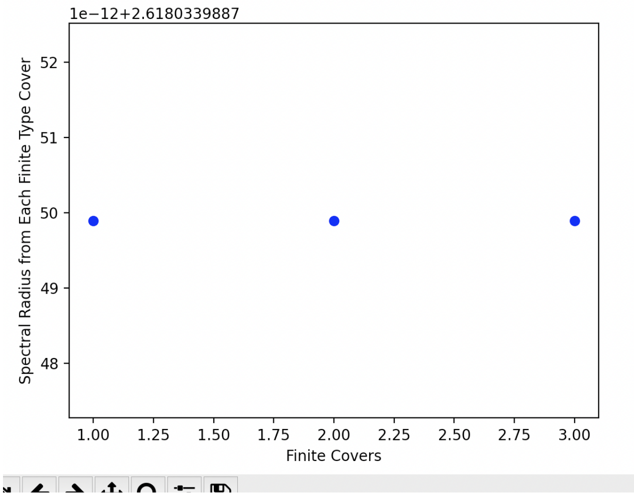
From the matrix, we see that the eigenvalues are the following: 2.61, $-0.5 \pm 0.866i$ and 0.382. Thus, the spectral radius for this cover is: 2.61.

8.5 Graph Results

In the last few sections, we have showed how the software operates to get the virtual homological spectral radius for each cover. Given the input n to determine the number of sheets of the cover, the program will give a graph which contains the values of the virtual spectral homological radius from each n-sheeted cover for which the radius exist. I will now show how the graph output is like for various n-sheeted covers.

4-sheeted covers of $\Sigma_{1,1}$, Pseudo-Anosov function $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

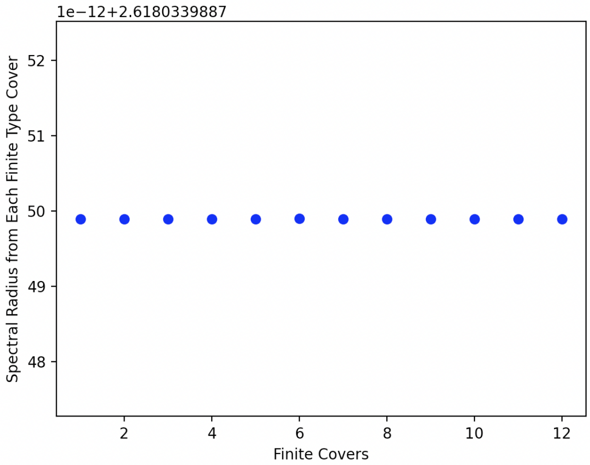
The total number of valid 4-sheeted covers is 3 out of a total 276. The remaining covers are not included as either: (i) the covers are not connected or (ii) the pseudo-anosov function does not lift to the cover. The spectral radii of the 4-sheeted covers are as follows:



Note that the values on the y-axis are read as n x $10^{12} + 2.618$, where n is the label on the left.

5-sheeted covers of $\Sigma_{1,1}$, Pseudo-Anosov function $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

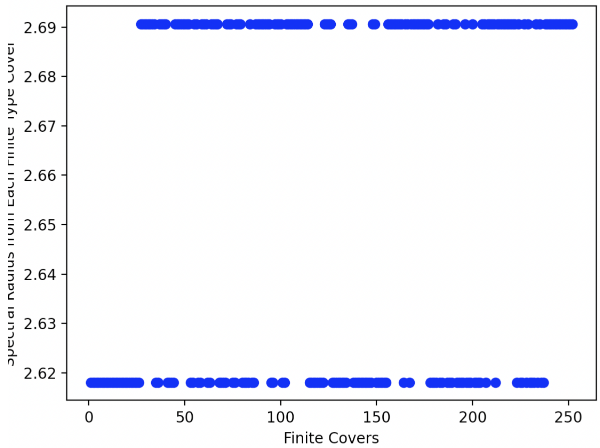
As mentioned above, the total number of valid 5-sheeted covers is 12 out of the total 7140 covers. This graph shows the following output:



Note that the values on the y-axis are read as n x $10^{12} + 2.618$, where n is the label on the left.

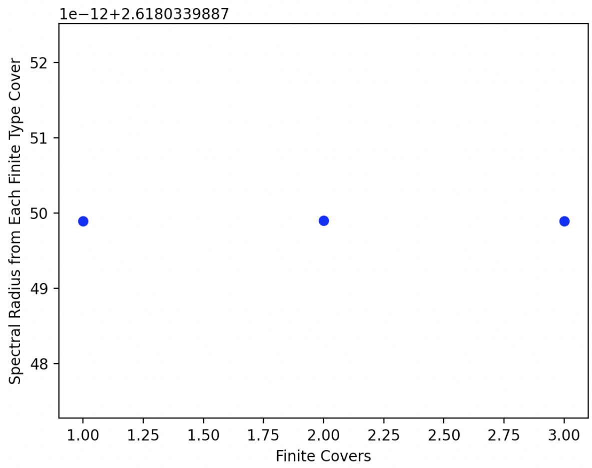
6-sheeted covers of $\Sigma_{1,1}$, Pseudo-Anosov function $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

As mentioned above, the total number of valid 6-sheeted covers is 250 out of the total 258840 covers. This graph shows the following output:



4-sheeted covers of $\Sigma_{1,1}$, Pseudo-Anosov function $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

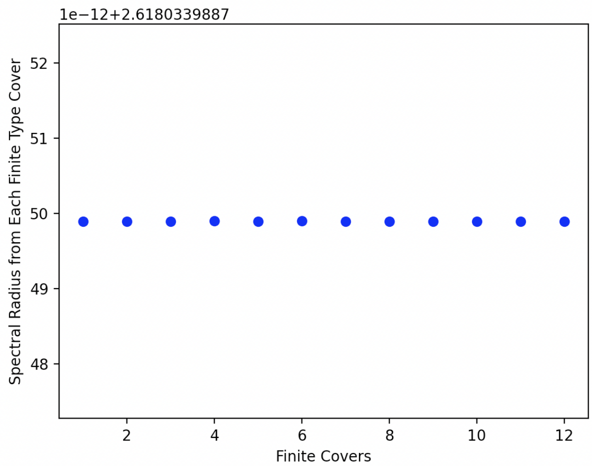
The total number of valid 4-sheeted covers is 3 out of a total 276:



Note that the values on the y-axis are read as n x $10^{12} + 2.618$, where n is the label on the left.

5-sheeted covers of $\Sigma_{1,1}$, Pseudo-Anosov function $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

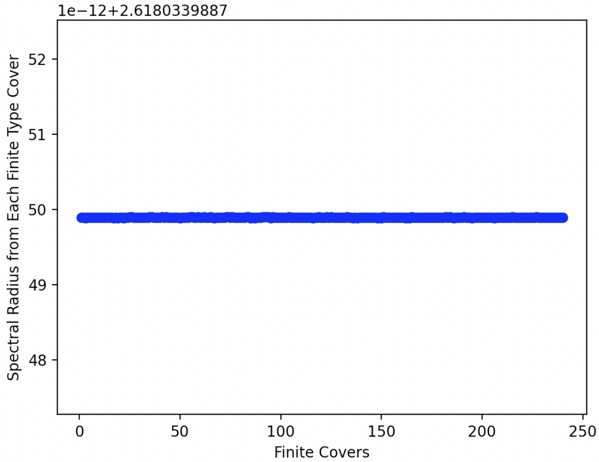
As mentioned above, the total number of valid 5-sheeted covers is 12 out of the total 7140 covers. This graph shows the following output:



Note that the values on the y-axis are read as n x $10^{12} + 2.618$, where n is the label on the left.

6-sheeted covers of $\Sigma_{1,1}$, Pseudo-Anosov function $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

As mentioned above, the total number of valid 6-sheeted covers is slightly less than 250 out of the total 258840 covers. This graph shows the following output:



Note that the values on the y-axis are read as n x $10^{12} + 2.618$, where n is the label on the left.

8.6 Interpreting the results

Dr Thang Le's conjecture states that:

$$log(D_h(f)) \geq \frac{Vol(M_f)}{3\pi|\chi|}.$$

Modifying Thang Le's general conjecture to our class of space, he proposes that:

$$\begin{aligned} log(D_h(f)) &\geq \frac{M_f}{3\pi|\chi|} \\ &= \frac{2.02988}{3\pi|1|} \\ &= \frac{2.02988}{3\pi} \\ &= 0.21537695725. \end{aligned}$$

Thus, we are required to show the following:

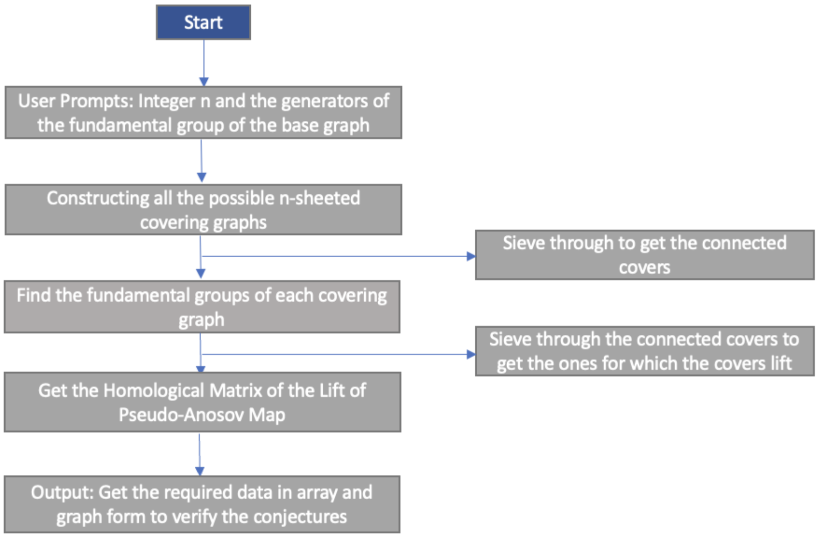
$$\begin{aligned} D_h(f) &\geq e^{0.21537695725} \\ &= 1.24032936. \end{aligned}$$

This can clearly be seen for the cases we have tested. Thus, we can see that the conjecture holds thus far, as w no counterexample has been found.

9 Logic Checks

9.1 Swiss Cheese Model of Checks

The software has inbuilt checks in different stages of the program to ensure that we do not have ridiculous outputs. Generally speaking, these are the stages in the software:



The checks include the following:

Checks involving Cardinality of Arrays Throughout the process, the software builds arrays to store information required whenever necessary. Key examples include arrays containing the pairs of permutation matrices to construct the covers, arrays containing the edges of the spanning tree of the cover, and arrays containing the fundamental group of each cover. In each of these cases, we can mathematically prove what the array size is supposed to be based on our inputs: the integer n, and the generators of the fundamental group of the base space.

In the first given example, we know that we need to construct a total of $\binom{n!}{2}$ covers, including covers which are not connected and covers for which the pseudo-Anosov function does not lift. When the software ends, it prints out the total number of covers it has considered at the start of the program. In the case of the 4, 4 and 6 sheeted covers, the software ensures that all 276, 7140 and 258840 covers are considered. By checking that the total number of covers we have constructed is equal to $\binom{n!}{2}$ covers, we ensure that no possible cover is missed.

In the second given example, the software stores the edges in the spanning tree in an array. We know that for any given spanning tree , the number of edges in the spanning tree is equal to the number of vertices in the spanning tree minus 1. Thus, in the software, as we know thus the number of vertices in the spanning tree is n, given by the input integer n, the program verifies that in the number of edges in the spanning tree is equal to $n-1$ for the spanning tree we find in each cover that we construct. This ensures that the correct spanning tree is built.

In the last example stated above: The generators of the fundamental group of each covering graph is also stored in the form of arrays in the software. As we learnt from the theorem constructing the generators of the covering graph, we know that the number of generators in the covering graph is equal to the total number of edges in the covering graph minus the number of edges in the spanning tree. The software is able to calculate the number of edges in the covering graph based on the user inputs: The construction by Monodromy Representation tells us that the number of edges in the covering graph is the number of vertices in the covering graph, n, multiplied by the number of generators of the fundamental group of the base space that the user inputs. So, in our case, as the base space is $\Sigma_{1,1}$, the number of generators is 2. Thus, the total number of edges in the covering graph is $2n$. The software has already calculated the number of edges in the spanning tree. Then, the software ensures that the size of the array containing the generators of the fundamental group of the covering graph is equal to the total number of edges in the covering graph minus number of edges in the spanning tree. So, in cases listed in the results, the software checks that the array containing the generators of the fundamental group of the covering graph is $2n-(n-1)=n+1$. This ensures that no generator is missed or we incorrectly add an extra generator.

Well-known Bounds There exist well known and studied bounds of the virtual homological spectral radius of pseduo-Anosov functions. As stated in the abstract, it is evident that any virtual homological spectral radius for f is greater than or equal to 1. It is shown by C.T. McMullen that any virtual homological spectral radius for a pseudo-Anosov surface automorphism f is strictly lesser than the dilation if the invariant foliations for f have prong singularities of odd order [McM13]. The software verifies this is true for each cover we construct, thus ensuring we do not have any anomalies in our data.

Each of this checks built into different stages of the software act as a barrier to ensure the integrity of our results. However, we can always add in additional layers and tighter bounds decrease the room of error. Due to the tight timeline in this project, these are the checks added in. This will be addressed in the scope of improvement section of the conclusion.

10 Conclusion and Future Scope of Improvement

While there are positive signs that the conjecture by Dr Thang Le holds, we thus far have only considered a specific class of surface and functions: 6-sheeted covers of the surface $\Sigma_{1,1}$ and pseudo-Anosov function $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Thus, the future direction primarily lies in expanding the cases of spaces and functions we can work on. More work can also be done to increase the speed and improve the memory management of the program, so that it becomes more efficient.

More surfaces we can work on: In this project, we specifically worked on the once punctured torus $\Sigma_{1,1}$. We can generalise this systematically.

First, we can focus on the case of the surfaces whose fundamental group is the free group. One such case is the bouquet of n circles, B_n. The reasons for focusing on this class of surfaces is simply because we can focus on generalising the construction of the cover of the base graph with n generators instead of just 2 without being hindered by the complexity of relations in the fundamental group.

Then, once we have coded the the construction of the cover of the base graph with n generators instead of just 2, we can then expand on this by introducing relations. At this stage, we need to remind ourselves that the base surface is no longer just a 1 cell complex. So, we need to figure out how to lift relations from the base surface to the covering surface and we also need to consider what are the impacts of relations of the base surface in the covering surface. We need to also introduce a new method to find the fundamental group of the covering surface as the method introduced in this thesis is only applicable for graphs and thus no longer relevant.

At the end of these 2 stages, we should be able to cover a large class of surfaces.

Generalising functions tested: To expand on the class of functions we need to test on, we need to be wary that eventually we require the hyperbolic volume of the mapping torus of the function. Thus, we should focus on functions whose mapping torus is well-known for easier testing.

At the sametime, we can include the feature to calculate the hyperbolic volume of the mapping torus of the function using the theory from [Pur20].

Improving Proof of Correctness: While there are multiple layers of checks added into the program, there can always be more checks added to increase the integrity of the results and the minimize the degree of error. One of the suggestions include adding tighter bounds that have been proven to be true into the program as an additional layer of security.

Increasing Speed: When tested in the author's computer, after $n = 6$, the software runs considerably slowly. To combat this, there are primarily 2 things that can be done.

Firstly, there is no structure added to the program so far. We can toy with adding Object-Oriented Programming or Functional Programming techniques to provide more structure, thus reducing redundancy in the code and improving speed. The latter can also increase the correctness of the program due to its type checking capabilities.

Secondly, the core algorithms and data structure can be improved, There are plenty of if and for loops in the code, which drastically increases the runtime of the algorithm. Moreover, only basic data structures, lists, arrays and dictionaries were used. This might have slowed down the software and options can be explored to see how and if we can change these things.

Another suggestion is to use techniques of high performance computing [Lan17]/ parallel programming in python [Pal14] to increase the speed of the program.

Memory Management: One of the key stages that takes up the memory of the program is the building of the permutation matrices to construct the covering graphs. As n increase from 3 to 6, the number of covers increase drasticallly: $15, 276, 7140, 258840$. Thus, for large n, we will face memory management issues. So, we should find ways to overcome this issue.

Bibliography

[Lic64] W. B. R. Lickorish. "A finite set of generators for the homeotopy group of a 2-manifold". In: *Proc. Cambridge Philos. Soc.* 60 (1964), pp. 769–778.

[Mum67] David Mumford. "Abelian quotients of the Teichmüller modular group". In: *J. Anal. Math* 18 (1967), pp. 227–244. DOI: 10.1007/BF02798046.

[Hum77] Stephen P. Humphries. "Generators for the mapping class group". In: *Topology of Low Dimensional Manifolds* 722 (1977), pp. 44–47.

[Pen88] Robert C Penner. "A Construction of pseudo-Anosov Homeomorphisms". In: *Trans. Amer. Math. Soc* 310 (1988), pp. 179–197.

[Sti93] John Stillwell. *Classical Topology and Combinatorial Group Theory*. Vol. 2. Graduate Texts in Mathematics. Springer, 1993, pp. 89–109. DOI: 10.1007/978-1-4612-4372-4.

[Mir95] Rick Miranda. *Algebraic Curves and Riemann Surfaces*. Vol. 5. Graduate Texts in Mathematics. American Mathematical Society, 1995, pp. 84–92. ISBN: 978-0-8218-0268-7.

[Thu98] W. Thurston. "Hyperbolic structures on 3–manifolds II: Surface groups and 3–manifolds which fiber over the circle". In: *preprint* - (1998), pp. -.

[Hat01] Allen Hatcher. *Algebraic Topology*. Walter de Gruyter, 2001.

[Mar12] Benson Farb Dan Margalit. *A Primer on Mapping Class Groups*. Princeton Mathematical Series. Princeton University Press, 2012.

[McM13] C. T. McMullen. "Entropy on Riemann Surfaces and the Jacobian of Finite Covers". In: *Commentarii Mathematici Helvetici* 88 (2013), pp. 953–964.

[Zie13] Gerhard Burde Heiner Zieschang. *Knots*. Cambridge University Press, 2013.

[Pal14] Jan Palach. *Parallel Programming with Python*. Graduate Texts in Mathematics. Packt Publishing, 2014.

[Mcs15] Sadayoshi Kojima Greg Mcshane. "Normalised Entropy versus Volume for Pseudo-Anosovs". In: *Geom. Topol. 22 (2018) 2403-2426* 22 (2015), pp. -.

[Lan17] Gabrielle Lanaro. *Python High Performance*. Vol. 2. Packt Publishing, 2017.

[Had20] Asaf Hadari. "Homological eigenvalues of lifts of pseudo-Anosov mapping classes to finite covers". In: *Geom. Topol.* 24 (2020). DOI: 10.2140/gt.2020.24.1717.

[Pur20] Jessica S. Purcell. "Hyperbolic Knot Theory". In: (2020). DOI: 10.48550/arXiv.2002.12652.

[Con] Keith Conrad. "SL(2,Z)". In: (). URL: https://kconrad.math.uconn.edu/blurbs/grouptheory/SL(2,Z).pdf.

[Deh10] Max Dehn. *Papers on Group Theory and Topology*. Princeton Mathematical Series. Springer-Verlag, 1909/1910. DOI: 10.1007/978-1-4612-4668-8.

11 Appendix

Here, I will show an example of the data output in my computer for the case $\Sigma_{1,1}$, $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

```
What do you want the size of n in S_n to be
4
What do you want the of generators for free group to be
a,b
The Psuedo-Anosov function is (In Penner Construction Form) {a: ['a', 'b', 'a'], b: ['a', 'b'], '-a': ['-a', '-b', '-a'], '-b': ['-b', '-a']}
  Row Number  rho                                                        Covering Graph Edges                      Spanning Tree Edges
------------  ---------------------------------------------------------  ----------------------------------------  ------------------------  --
         149  {'a': array([[0., 1., 0., 0.],                             [(2, 1), (3, 4), (3, 1), (4, 3), (4, 2)]  [(1, 2), (1, 3), (2, 4)]
                     [1., 0., 0., 0.],
                     [0., 0., 0., 1.],
                     [0., 0., 1., 0.]]), 'b': array([[0., 0., 1., 0.],
                     [0., 0., 0., 1.],
                     [1., 0., 0., 0.],
                     [0., 1., 0., 0.]])}
         156  {'a': array([[0., 1., 0., 0.],                             [(2, 1), (3, 4), (3, 2), (4, 3), (4, 1)]  [(1, 2), (1, 4), (2, 3)]
                     [1., 0., 0., 0.],
                     [0., 0., 0., 1.],
                     [0., 0., 1., 0.]]), 'b': array([[0., 0., 0., 1.],
                     [0., 0., 1., 0.],
                     [0., 1., 0., 0.],
                     [1., 0., 0., 0.]])}
         255  {'a': array([[0., 0., 1., 0.],                             [(2, 4), (2, 3), (3, 1), (4, 2), (4, 1)]  [(1, 3), (1, 4), (3, 2)]
                     [0., 0., 0., 1.],
                     [1., 0., 0., 0.],
                     [0., 1., 0., 0.]]), 'b': array([[0., 0., 0., 1.],
                     [0., 0., 1., 0.],
                     [0., 1., 0., 0.],
                     [1., 0., 0., 0.]])}
Total number of covering spaces is: 277
The set consisting of spectral eigenvalues, one from each finite cover: [2.6180339887498936, 2.618033988749895, 2.6180339887498922]
D_h(f) is: 2.618033988749895
The logarithm of D_h(f) is: 0.9624236501192069
```

Number of Edges in Covering Graph but not Spanning Tree	List of paths in spanning tree from base point 1	Spanning Tree Paths as Words
3	[[1, 2], [1, 3], [1, 2, 4]]	[['a'], ['b'], ['a', 'b']]
3	[[1, 2], [1, 2, 3], [1, 4]]	[['a'], ['a', 'b'], ['b']]
3	[[1, 3, 2], [1, 3], [1, 4]]	[['a', 'b'], ['a'], ['b']]

```
Generators_for_Fundamental_Group_of_Covering_Graph                                  Test to see if all edges not in spanning tree is accounted for
----------------------------------------------------------------------------------  ----------------------------------------------------------------
[['a', 'a'], ['b', 'a', '-b', '-a'], ['b', 'b'], ['a', 'b', 'a', '-b'], ['a', 'b', 'b', '-a']]  Yes



[['a', 'a'], ['a', 'b', 'a', '-b'], ['a', 'b', 'b', '-a'], ['b', 'a', '-b', '-a'], ['b', 'b']]  Yes



[['a', 'b', 'a', '-b'], ['a', 'b', 'b', '-a'], ['a', 'a'], ['b', 'a', '-b', '-a'], ['b', 'b']]  Yes
```

```
Where does the pseudo-anosov map send the generators of the fundamental group of the covering space to?
---------------------------------------------------------------------------------------------------------------------------------------------------------------------
[['a', 'b', 'a', 'a', 'b', 'a'], ['a', 'b', 'a', 'b', 'a', '-b', '-a', '-a', '-b', '-a'], ['a', 'b', 'a', 'b'], ['a', 'b', 'a', 'a', 'b', 'a', 'b', 'a', '-b', '-a'],


[['a', 'b', 'a', 'a', 'b', 'a'], ['a', 'b', 'a', 'a', 'b', 'a', 'b', 'a', '-b', '-a'], ['a', 'b', 'a', 'a', 'b', 'a', 'b', '-a', '-b', '-a'], ['a', 'b', 'a', 'b', 'a'


[['a', 'b', 'a', 'a', 'b', 'a', 'b', 'a', '-b', '-a'], ['a', 'b', 'a', 'a', 'b', 'a', 'b', '-a', '-b', '-a'], ['a', 'b', 'a', 'a', 'b', 'a'], ['a', 'b', 'a', 'b', 'a'
```

```
                                                        Can the function be lifted?    Homological Matrix                   Homological Matrix without spanning tree edges
-------------------------------------------------------  ----------------------------  -----------------------------------  ----------------------------------------------
 ['a', 'b', 'a', 'a', 'b', 'a', 'b', '-a', '-b', '-a']]  This passes the test          [[ 1.  1.  0.  1.  1.  1.  0.  1.]   [[ 1.  1.  0.  1.  1.]
                                                                                        [ 0. -1.  1.  0. -1.  1.  0.  0.]    [ 0. -1.  1.  0. -1.]
                                                                                        [ 0.  0.  1.  1.  0.  1.  0.  1.]    [ 0.  0.  1.  1.  0.]
                                                                                        [ 1.  2.  0.  1.  1.  0.  1.  0.]    [ 1.  2.  0.  1.  1.]
                                                                                        [ 1.  1.  0.  0.  1.  0.  1.  0.]]   [ 1.  1.  0.  0.  1.]]

', '-b', '-a', '-a', '-b', '-a'], ['a', 'b', 'a', 'b']]  This passes the test          [[ 1.  1.  1.  1.  0.  1.  0.  1.]   [[ 1.  1.  1.  1.  0.]
                                                                                        [ 1.  1.  1.  2.  0.  0.  1.  0.]    [ 1.  1.  1.  2.  0.]
                                                                                        [ 1.  0.  1.  1.  0.  0.  1.  0.]    [ 1.  0.  1.  1.  0.]
                                                                                        [ 0.  0. -1. -1.  1.  1.  0.  0.]    [ 0.  0. -1. -1.  1.]
                                                                                        [ 0.  1.  0.  0.  1.  1.  0.  1.]]   [ 0.  1.  0.  0.  1.]]

', '-b', '-a', '-a', '-b', '-a'], ['a', 'b', 'a', 'b']]  This passes the test          [[ 1.  1.  1.  2.  0.  0.  1.  0.]   [[ 1.  1.  1.  2.  0.]
                                                                                        [ 0.  1.  1.  1.  0.  0.  1.  0.]    [ 0.  1.  1.  1.  0.]
                                                                                        [ 1.  1.  1.  1.  0.  1.  0.  1.]    [ 1.  1.  1.  1.  0.]
                                                                                        [ 0. -1.  0. -1.  1.  1.  0.  0.]    [ 0. -1.  0. -1.  1.]
                                                                                        [ 1.  0.  0.  0.  1.  1.  0.  1.]]   [ 1.  0.  0.  0.  1.]]
```

```
eigenvalues                                                          Absolute Value of Eigenvalues
-------------------------------------------------------------------  ----------------------------------------------------------------------------------------------------
[ 2.61803399+0.j        -0.5       +0.8660254j -0.5       -0.8660254j  [2.6180339887498936, 1.0000000000000004, 1.0000000000000004, 0.9999999999999997, 0.3819660112501052]
  1.        +0.j         0.38196601+0.j       ]



[ 2.61803399+0.j        -0.5       +0.8660254j -0.5       -0.8660254j  [2.618033988749895, 0.9999999999999999, 0.9999999999999999, 1.0000000000000002, 0.3819660112501051]
  1.        +0.j         0.38196601+0.j       ]



[ 2.61803399+0.j        -0.5       +0.8660254j -0.5       -0.8660254j  [2.6180339887498922, 1.000000000000001, 1.000000000000001, 1.0, 0.38196601125010515]
  1.        +0.j         0.38196601+0.j       ]
```