Thurston's Construction

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1 Introduction

In this article, I will be introducing Thurston's Construction. We know that Nielsen-Thurston Classification tells us that given a surface $\Sigma_{g,n}$, with $g, n \geq 0$, every mapping class $f \in \text{Mod}(\Sigma_{g,n})$ is periodic, reducible or pseudo-Anosov. Then, the question arises: given an arbitary mapping class f, what classification does it fall under? Thurston's construction addresses this question. Briefly put, Thurston's Construction tells us that we can classify a mapping class based on the action of the multitwists that generate the mapping class group (which we will define later) on the equivalence classes of the frame fields of the Singular Euclidean Structure of $\Sigma_{g,n}$.

Before giving the formal statement, let's look at a few relevant definitions. Let $S = \Sigma_{g,n}$. We say that a collection of isotopy classes of simple closed curves in S fills S if any simple closed curve in S has positive geometric intersection with some isotopy class in the collection. If $A = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a multicurve in a surface S (that is a set of pairwise disjoint simple closed curves), we denote the product $\Pi_{i=1}^{n} T_{\alpha_i}$ by T_A. Such a mapping class is often called a multitwist.

The formal statement of the theorem is as follows:

Theorem 1.1 (Thurston's Construction)**.** *Suppose A and B are multicurves in S, so that $A \cup B$ fills S. There is a real number $\mu = \mu(A, B)$ and a representation $\rho :< T_A, T_B > \to PSL(2, \mathbb{R})$ given by:*

$$T_A \mapsto \begin{pmatrix} 1 & -\mu^{\frac{1}{2}} \\ 0 & 1 \end{pmatrix} \text{ and } T_B \mapsto \begin{pmatrix} 1 & 0 \\ \mu^{\frac{1}{2}} & 1 \end{pmatrix}.$$

The representation ρ has the following properties:

A. *An element $f \in < T_A, T_B >$ is periodic, reducible or pseudo-Anosov according to whether $\rho(f)$ is elliptic, parabolic or hyperbolic.*

B. *When $\rho(f)$ is parabolic, f is a multitwist.*

C. *When $\rho(f)$ is hyperbolic, the stretch factor of the pseudo-Anosov mapping class f is equal to the larger of the 2 eigenvalues of $\rho(f)$.*

Proof. The general idea for proving the theorem is to find a singular Euclidean Structure on S with respect to which any function $\phi \in < T_A, T_B >$ acts by affine transformation. The function $\phi \in < T_A, T_B >$ needs to act by affine transformation in order to respect the foliation-preserving property that the surface itself possesses (leaves in a foliation are sent to leaves). Here, an affine transformation is one that, in local charts away from singularities, is of the form $Mx + b$, where M is a linear transformation and b is a vector. In Thurston's Work on Surfaces, Thurston introduces an equivalent definition of affine transformation: A homeomorphism ϕ is said to be affine if it leaves invariant the set of co-vertices (which is defined later) and if the image of a straight line of the flat structure is a straight line.

We will then add an additional feature that comes equipped with an orthonormal frame field, well defined up to sign. A local frame field for a manifold M (defined on $p \in U \subset M$) is a collection $(X_1, \ldots, X_n)$ of smooth vectors fields that are defined on an open neighborhood $U \subset M$ such that for each $q \in U$, the collection $(X_1(q), \ldots, X_n(q))$ is a basis of $T_q(M)$ and $\forall i, X_i(q)$ is orthogonal and have length 1. In our case, as the singular euclidean structure is embedded in $\mathbb{R}^2$, the local frame field only consists of 2 smooth vector fields X_1, X_2. Then, any given affine map on S, its derivative can be described by a 2 x 2 matrix well defined up to sign: The representation ρ will assign to each affine map in $< T_A, T_B >$ its differential, $\rho(h) = Dh$.

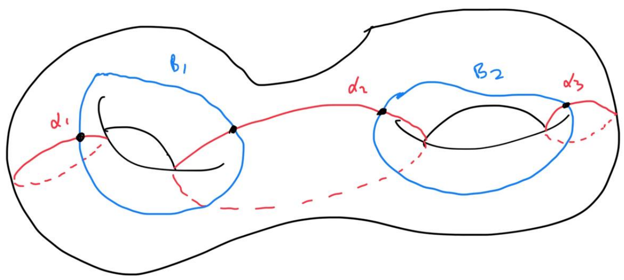
So, the proof of this construction breaks down into the following steps:

- Decompose S into its Singular Euclidean Structure.
- Once the singular Euclidean structure on which T_A and T_B act affinely has been established, after orienting multicurves A and B, we will give the structure an orthogonal frame field well defined up to multiplication by ± 1: We choose a positively orientated basis so that the first vector is parallel to A and the second vector is parallel to B.
- By construction, in this singular Euclidean Structure, the multitwists T_A and T_B can be chosen to be affine. These affine transformations, T_A and T_B, fix the 1-cells of C parallel to A and B, respectively. The action of T_A and T_B on equivalence classes of frame fields are then given exactly by the matrices in the statement of the theorem.
- We now finish the proof by showing the representation satisfies the properties stated in the theorem.

I will now explicitly break down each step.

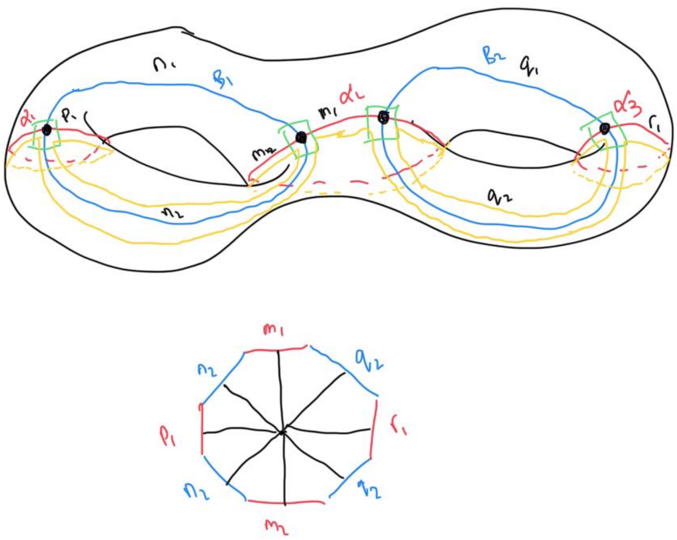
- Decompose S into its measured foliation as stated in the above section.

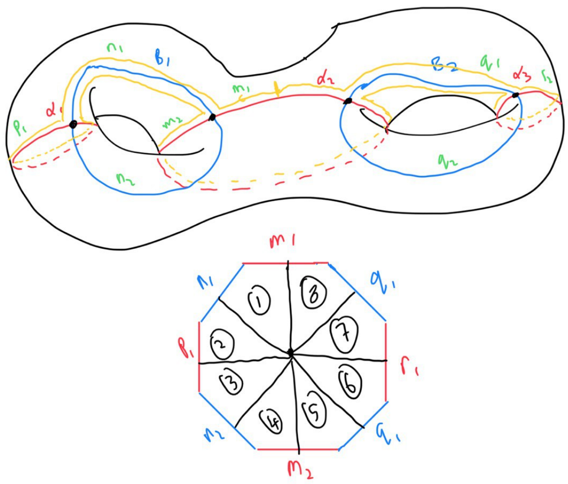
We will decompse S into a flat metric as follows. As the process is quite involved, we will use the surface $\Sigma_{2,0}$. The red and blue curves form the set of multicurves $T_A = \{\alpha_1, \alpha_2, \alpha_3\}$ and $T_B = \{\beta_1, \beta_2\}$. The set of curves $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\}$ fills $\Sigma_{2,0}$.



Our endgoal: Since our example of $\Sigma_{2,0}$ does not contain any singular point, we are simply trying to construct a flat representation (embedding) of the surface on $\mathbb{R}^2$ such that parallel lines remain parallel under the action of T_A and T_B (as T_A and T_B are supposed to be leaf-preserving).

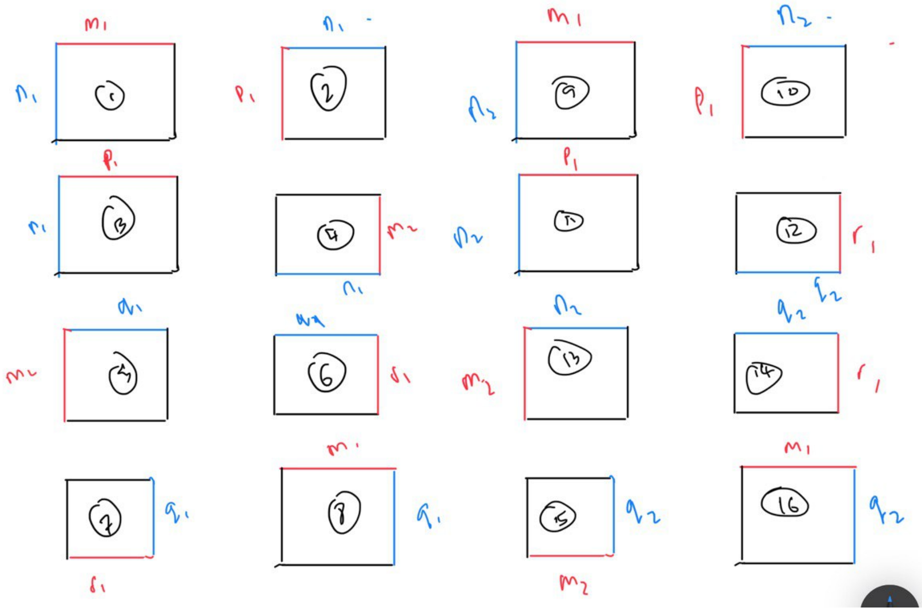
We can think of $\alpha_i \cup \beta_j$ as a 4-valent graph (that is to say, the number of edges that are incident to the vertex is 4) in $\Sigma_{2,0}$, where the vertices are the points of $\alpha \cap \beta$ (represented by the black vertices above in the diagram). In fact, by also considering the closures of the components of $S - \alpha \cup \beta$ as 2-cells, we have a description of S as a 2-complex X. By cutting the surface along the 1-complex $T_A \cup T_B$, we can decompose $\Sigma_{2,0}$ into 2 2-cell complexes.



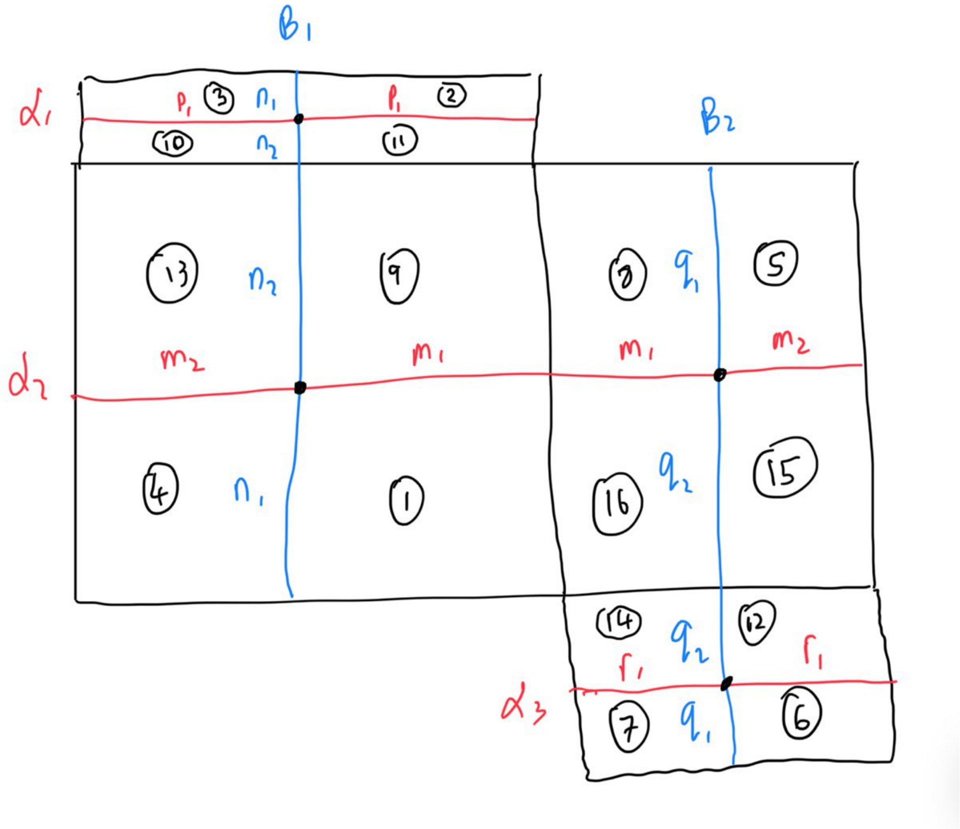


We construct a dual complex X' of $\Sigma_{2,0}$ using the 2 2-cell complexes . This complex is formed by taking one vertex for each 2-cell of X, called the co-vertex, one edge transverse to each edge of X, called co-edges, and one 2-cell for each vertex of X, called co-cells. If the 2-cell has a puncture or a marked point in it, then the marked point/puncture will be the co-vertex. This is also shown above.

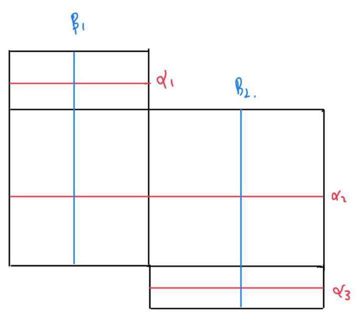
In our case, as our 2-cell is an octagon with 8 edges protruding from the co-vertex, we further decompose our 2 co-cells into 16 rectangles. The case for the co-cell in the above picture is shown below.



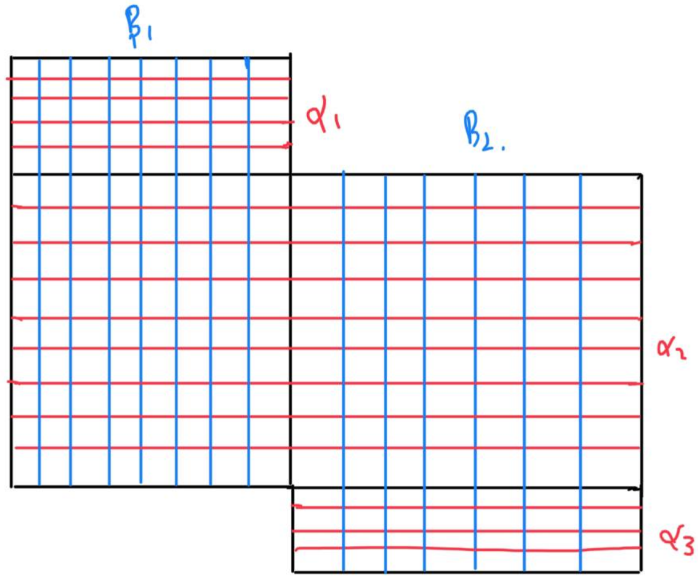
We can glue these squares together to form a bunch of rectangles which can be embedded in $\mathbb{R}^2$.



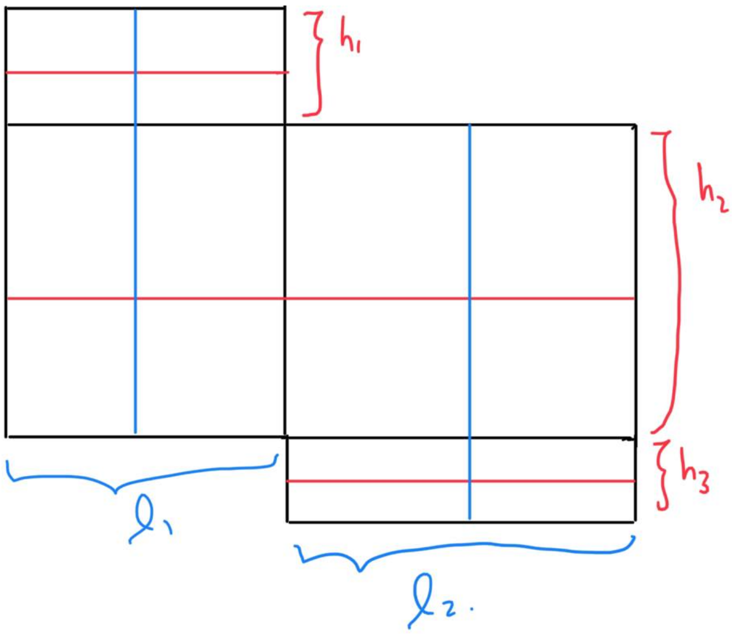
Since the vertices of X are 4-valent, it follows that X' is a square complex, that is each 2-cell of X' is a square. What is more, each square of X' has a segment of α running from one side to the opposite side.



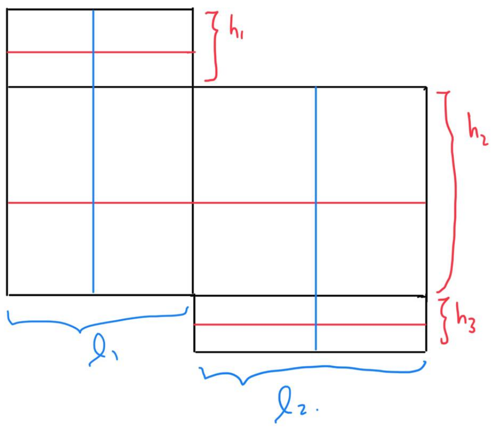
We can foliate each square of X' by lines parallel to α. This gives rise to a foliation F_α on all of S.



In order to embed the rectangles into $\mathbb{R}^2$, we need to provide the rectangels a geometry. Ie, we need to declare the width of each square to be the same fixed number, and this gives a measure on F_α. The foliation associated to β is a measured foliation F_β that is transverse to F_α.

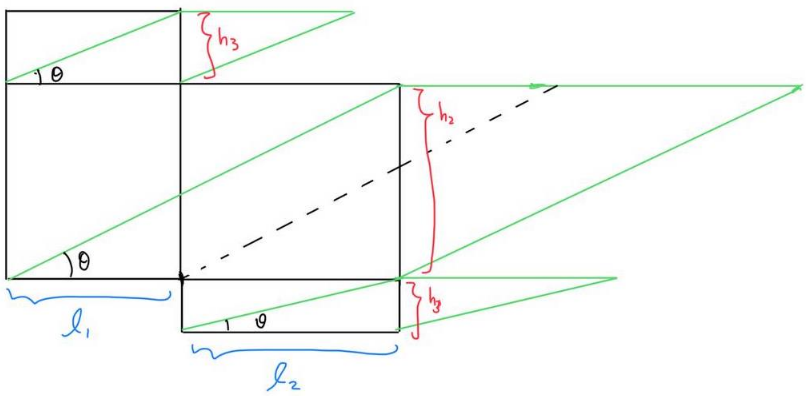


Embedding of $\Sigma_{2,0}$ into $\mathbb{R}^2$: The first step of constructing the representation in Thurston's construction involves embedding $\Sigma_{2,0}$ into $\mathbb{R}^2$. While half the work was done in the previous section decomposing $\Sigma_{2,0}$ into a flat structure (rectangles), we have yet to give geometry to these rectangles. That is to say we have to specify the lengths of h_1, h_2, h_3, l_1 and l_2:

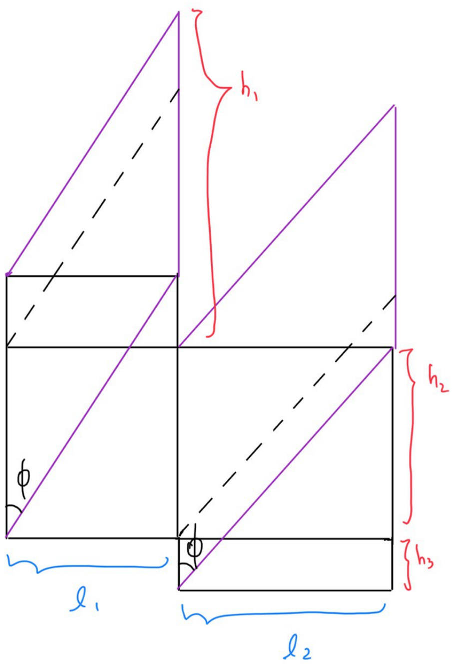


As we will see further down, this boils down to solving a system of linear equations.

Note that as T_A is a Dehn twist on non-intersecting curves α_1, α_2, and α_3, the action of T_A on the rectangle is as follows:



Note that as T_B is a Dehn twist on non-intersecting curves β_1 and β_2, the action of T_B on the rectangle is as follows:



2 key facts that will give us the system of linear equations to solve are:

- We want T_A and T_B to act affinely on the rectangles in $\mathbb{R}^2$. This means that we want parallel lines to remain parallel after the transformation.

- So, T_A and T_B act affinely if and only if the slopes of the rectangles are constant after T_A and T_B act affinely on the rectangles.

This results in the following sets of equations:

$$\lambda = tan(\theta) = \frac{h_1}{l_1} = \frac{h_2}{l_1 + l_2} = \frac{h_3}{l_2}$$
$$\mu = tan(\phi) = \frac{l_1}{h_1 + h_2} = \frac{l_2}{h_2 + h_3}$$

Using the fact that $v = \mu\lambda$, we get the following pair of system of linear equations, one dealing with variables h_i and the other dealing with l_j.

The length equations:

$$\frac{l_1}{v} = 2l_1 + l_2$$
$$\frac{l_2}{v} = l_1 + 2l_2$$

The height equations:

$$\frac{h_1}{v} = h_1 + h_2$$
$$\frac{h_2}{v} = h_1 + 2h_2 + h_3$$
$$\frac{h_3}{v} = h_2 + h_3$$

When the equations above are represented in linear algebra form:

$$\frac{1}{v}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$
$$\frac{1}{v}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

Then, we can see that the eigenvectors of the equations above, give us the dimensions for the sides of the rectangles. In order for Thurston's Construction to work, we will be require a specfici choice among the possibilies, which I will elaborate on further. After obtaining the lengths and the height we can embed $\Sigma_{2,0}$ into $\mathbb{R}^2$.

1.1 Generalising the Embedding of Co-cells into $\mathbb{R}^2$

While we can reduce the problem in the above scenario to simple sets of system of linear equation, this might not always be easy. However, there is an important fact that allows us to skip the process of obtaining the length and height equations via trigonometry and immediately obtain the matrix form: It can be shown that the choice for h_i and l_j can be obtained from the following matrix:

$$N = \begin{pmatrix} i(\alpha_1, \beta_1) & i(\alpha_1, \beta_2) \\ i(\alpha_2, \beta_1) & i(\alpha_2, \beta_2) \\ i(\alpha_3, \beta_1) & i(\alpha_3, \beta_2) \end{pmatrix}.$$

To verify, note that:

$$NN^t = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ and}$$
$$N^tN = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Thus, for the general case with the multicurves $A = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $B = \{\beta_1, \beta_2, ..., \beta_n\}$. Let N be the matrix with (j, k) entry, we define the matrix N as

$$N_{j,k} = i(\alpha_j, \beta_k).$$

Theorem 1.2. *N is primitive.*

Proof. A non-negative matrix is primitive if it has a power that is a positive matrix. A matrix is said to be positive (non-negative) if each of its entries is positive (non-negative).

Given N, let G be the abstract bipartite graph with m red vertices and n blue vertices, and $N_{j,k}$ edges between the jth vertex and the kth blue vertex. Then, the (j, k) entry of the dth power $(NN^t)^d$ is equal to the number of paths in G of length $2d$ between the jth and kth red vertices in G. Indeed, this is equivalent to the statement that the graph G is connected. If G is not connected, that would mean that $A \cup B$ is not connected, and so the pair A, B does not fill the surface. Thus, N is primitive.

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Then, we can use the perron-frobenius theorem:

Theorem 1.3 (Perron-Frobenius matrices)**.** *Let A be an n x n matrix with integer entries. If A is a primitive, then A has a unique nonnegative unit eigenvector v. The vector v is positive and has a positive eigenvalue that is larger in absolute value than all other eigenvalues.*

which tells us we can find the vectors we require. The vectors give us the possible sets of the width and length for the rectangles, for which T_A and T_B act affinely, so that we can embed them into the real plane.

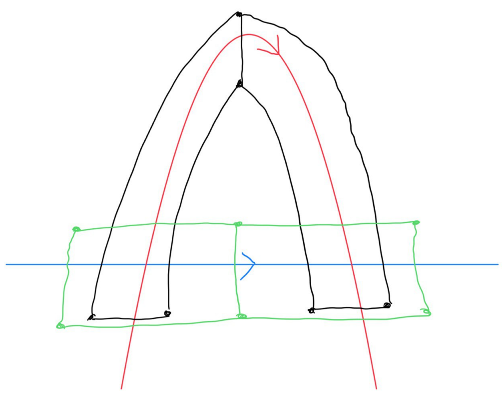
Going back to the point in the previous section, among the possible widths and lengths we can choose from, we will choose the values such that:

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = (\frac{1}{v})^{-\frac{1}{2}} N^t \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

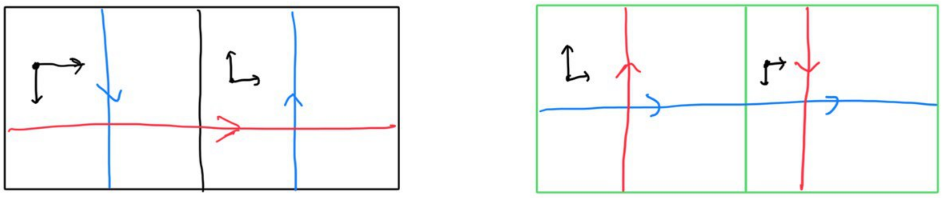
Once we have assigned the lengths and widths to the rectangles, we can embed the structure into $\mathbb{R}^2$.

- Once the singular Euclidean structure on which T_A and T_B act affinely has been established, after orienting multicurves A and B, we will give the structure an orthogonal frame field well defined up to multiplication by ± 1: We choose a positively orientated basis so that the first vector is parallel to A and the second vector is parallel to B.

To see the ambiguity of ± 1, we can consider the following situation.



(1)



(2)

We can see breaking the curves into the cocells, the orthonormal frame fields might differ in orientation by 180 degrees, even though the foliations are derived from the same curve. Thus, we need to ± 1.

- By construction, in this singular Euclidean Structure, the multitwists T_A and T_B can be chosen to be affine. These affine transformations, T_A and T_B, fix the 1-cells of C parallel to A and B, respectively. The action of T_A and T_B on equivalence classes of frame fields are then given exactly by the matrices in the statement of the theorem.

- We now finish the proof by showing the representation satisfies the properties stated in the theorem.

If $\rho(f)$ is elliptic, then $p(f)$ has a power such that add more details here. So, f has a power that fixes the orthogonal frame field of S (up to sign) at every point. Also, by construction, f fixes each singular point of the metric. Thus, f has a power that acts as the identity in the neighborhood of some singular point. Since f is affine, it follows that f is periodic.

If $\rho(f)$ is parabolic, then it has a 1-dimensional eigenspace, and the eigenvalue for this space must be 1. The eigenspace induces a singular foliation on S. Up to replacing f with a power, we may assume that (the affine representative of) f fixes each singularity of the foliation and preserves each leaf emanating from each singularity. Let L be one such leaf. Since the eigenvalue is 1, it follows that f fixes L pointwise. If the leaf L had an acccumulation point, then it would follow that f fixes a neighborhood of this accumulation point, and so (a power of) f would be the identity. Thus, we may assume that the collection of all leaves starting from singular points is a collection of closed curves in S. As these closed curves are geodesics in the singular Euclidean metric, they are all simple and homotopically non-trivial. Since f fixes this collection, it follows that f is reducible. What is more, if we cut S along the reducing curves, we obtain a foliation that does not have any singularities. By the Euler-Poincare formula, the cut surface must be a collection of annuli. In particular, f is a multitwist about reducing curves.

Finally, if $\rho(f)$ is hyperbolic, then the eigenspaces of $\rho(f)$ define 2 transverse measured foliations, f multiplies the measure of one foliation by the larger eigenvalue of $\rho(f)$, and f multiplies the measure of the other foliation by the smaller eigenvalue of $\rho(f)$ (the foliations have singularities at the singular points of the Euclidean structure). Thus, f is pseudo-Anosov, and its stretch factor is given by the larger eigenvalues of $\rho(f)$.

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