A model-informed neural network for solving inverse scattering problem

Aravinth Krishnan

Kansas State University

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Outline

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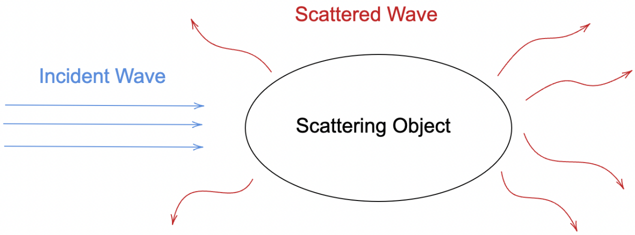
Introduction



Inverse scattering problem: Determine the *scattering object* from boundary measurements of scattered wave (for several incident waves).

Applications: Radar, non-destructive testing, geophysical exploration, medical imaging, ...

Inverse scattering problem

- Let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be bounded function satisfying $\eta = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Let Ω be large disk such that $D \subset \Omega$. Consider incident waves

$$u^{in}(x, d) = e^{ikd\cdot x}$$

where $d \in S := \{x \in \mathbb{R}^2 : |x| = 1\}$.

- Consider the following model problem

$$\begin{cases} \Delta u + k^2(1 + \eta(x))u = 0 & \text{in } \mathbb{R}^2 \\ u = u^{in} + u^{sc} \\ \lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^{sc}}{\partial r} - ik u^{sc}\right) = 0, & r = |x|. \end{cases}$$

- **Inverse Problem:** **Given** $u^{sc}(\cdot, d)|_{\partial\Omega}$ for all $d \in S$, **determine** η.

Deep Learning for inverse scattering

Supervised learning-based algorithms (far-from-complete list)

- Z. Wei and X. Chen, *Deep-learning schemes for full-wave nonlinear inverse scattering problems*, IEEE Trans. Geosci. Remote Sens., 57 (2019) 1849–1860.
- Y. Khoo and L. Ying, *SwitchNet: a neural network model for forward and inverse scattering problems*, SIAM J. Sci. Comput., 41 (2019) A3182-A3201.
- Y. Sanghvi, Y. Kalepu, and U. Khankhoje, *Embedding deep learning in inverse scattering problems*, IEEE Trans. Comput. Imaging, 6 (2020) 46–56.
- X. Chen, Z. Wei, L. Maokun, P. Rocca, *A review of deep learning approaches for inverse scattering problems*, Prog. Electromagn. Res., 167 (2020) 67–81.
- Y. Gao, H. Liu, X. Wang, and K. Zhang, *On an artificial neural network for inverse scattering problems*, J. Comput. Phys., 448 (2022) 110771.
- M. Zhou, J. Han, M. Rachh, and C. Borges, *A neural network warm-start approach for the inverse acoustic obstacle scattering problem*, J. Comput. Phys., 490 (2023) 112341.
- ...

Reconstruct unknown scattering objects that are similar or closely related to known training datasets!

Deep Learning for inverse scattering

Unsupervised learning: Physics Informed Neural Networks (PINNs)

- M. Raissi, P. Perdikaris, and G. E. Karniadakis, *Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear PDEs*, J. Comput. Phys. 378 (2019) 686-707.
- Y. Chen, L. Lu, G. E. Karniadakis, and L. Dal Negro, *Physics-informed neural networks for inverse problems in nano-optics and metamaterials*, Optics Express 28 (2020) 11618-11633.
- Y. Chen and L. Dal Negro, *Physics-informed neural networks for imaging and parameter retrieval of photonic nanostructures from near-field data*, APL Photonics 7 (2022) 010802.
- A. Pokkunuru, P. Rooshenas, T. Strauss, A. Abhishek, T. Khan, *Improved training of physics-informed neural networks using energy-based priors: a study on electrical impedance tomography*, ICLR 2023 (20 pages).

Study inverse design problems with "internal" data!

Our Algorithm

We developed a 2-step **unsupervised** deep learning algorithm to solve the inverse scattering problem:

- Step 1: Convert the boundary scattering data into an imaging function $I(z)$ which encodes geometrical information about η.
- Step 2: Use $I(z)$ as input for training. A model equation for $I(z)$ and η is employed to train a neural network which predicts η.

Outline

Stable Imaging Function

- For $z \in \mathbb{R}^2$, define the function

$$I(z) := \int_S \int_{\partial\Omega} u^{sc}(x, d)\overline{\Phi(x, z)}\, ds(x)\, \Phi^\infty(d, z)\, ds(d),$$

where $\Phi(x, z)$ and $\Phi^\infty(d, z)$ denote the Green's function and its scattering amplitude respectively.

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Theorem 1

The imaging function satisfies

$$I(z) = \frac{k\pi}{2} \int_D \left[J_0^2(k|y-z|) + J_0(k|y-z|) \int_S u^{sc}(y,d)\Phi^\infty(d,z) ds(d) \right] \eta(y)\, dy$$

Proof: Helmholtz-Kirchoff identity and Funk-Hecke formula.

Stable Imaging Function

We assume the noisy data u_δ^{sc} satisfies

$$\|u^{sc} - u_\delta^{sc}\|_{L^2(\partial\Omega\times\mathbb{S})} \leq \delta\|u^{sc}\|_{L^2(\partial\Omega\times\mathbb{S})}.$$

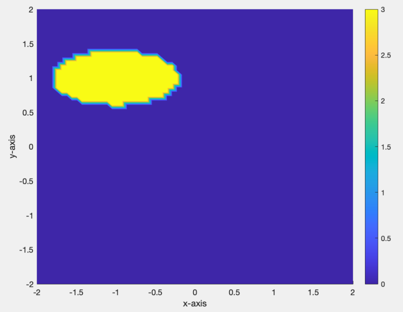
Theorem 2

Let $I_\delta(z)$ the imaging function with noisy data u_δ^{sc}. Then

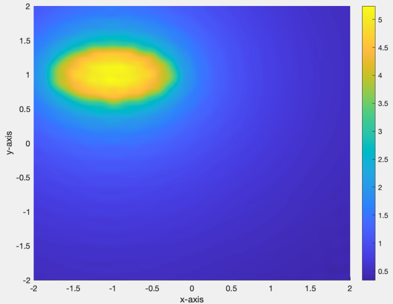
$$|\mathcal{I}(z) - \mathcal{I}_\delta(z)| \leq C\delta, \quad \text{for all } z \in \mathbb{R}^2,$$

where C is a positive constant which is independent of z and δ.

An Example



(a) True profile



(b) $I(z)$ with 20% synthetic noise in data

Only geometrical information (location and shape) about the target is reconstructed!

Reconstruction resolution is within the diffraction limit!

A Model Equation

Under the Born approximation, we can derive

$$I(z) - \frac{k\pi}{2} \int_D J_0^2(k|y-z|)\,\eta(y)\,dy = 0.$$

- A simple and nice connection between $I(z)$ and η.
- This equation will be used as a (simplified) model equation in training our neural network with data $I(z)$.

Outline

Model Informed Neural Network

Key Ideas:

- $I(z)$ can be obtained robustly and inexpensively from (noisy) scattering data.
- Use input data $I(z)$ and the model equation to train neural networks I_Θ and η_Θ to predict η.

Model Informed Neural Network

Let B be the domain in which we compute η. Given data $I(z)$, we train I_Θ and η_Θ with the model equation

$$I_\Theta(z) - \frac{k\pi}{2} \int_B J_0^2(k|y-z|)\,\eta_\Theta(y)\,dy = 0.$$

We consider the minimization problem

$$\min_\Theta \{MSD(\Theta) + MSM(\Theta)\},$$

where

$$MSD(\Theta) := \frac{1}{N} \sum_{j=1}^{N} |I_\Theta(z_j) - I(z_j)|^2,$$

$$MSM(\Theta) := \frac{1}{M} \sum_{j=1}^{M} \left| I_\Theta(z_j) - \frac{k\pi}{2} \int_B J_0^2(k|y-z_j|)\,\eta_\Theta(y)\,dy \right|^2.$$

Visual Representation

Let $z = (x, y)^\top \in \mathbb{R}^2$.

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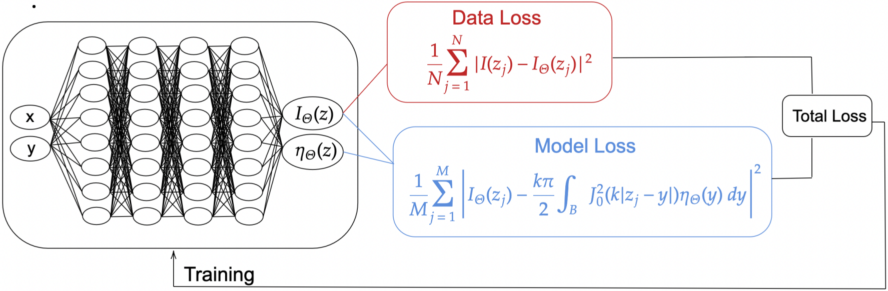

Figure 2: Visual representation of the model-informed neural network

Model Informed Neural Network

We consider a fully connected neural network:

- Number of Neurons in Input Layer $= 2$
- Number of Hidden Layers $= 4$
- Number of Neurons in each Hidden Layer $= 64$
- Number of Neurons in Output Layer $= 3$
- Activation Function in Hidden Layers $\rightarrow$ Hyperbolic Tangent
- Activation Function in Output Layers $\rightarrow$ Identity Function
- Kernel Initializer $=$ Glorot Normal
- Learning Rate: 0.001
- Optimizer: Adam
- Number of Iterations: $150,000$

Numerical study

Consider

- Computational domain $(-2,2)^2$ with 64×64 uniform grid points
- Wave number $k = 15$
- Measurement boundary $\partial\Omega$ is circle of radius 100 centered at $(0,0)^\top$.
- Synthetic additive noise model:

$$u^{sc} + \delta \frac{\mathcal{N}}{\|\mathcal{N}\|} \|u^{sc}\|,$$

  where δ is the noise level and $\mathcal{N}$ is noise matrix consisting of random entries $a + bi$ for $a, b \in (-1, 1)$

Example 1: Square profile, $\eta = 0.3$

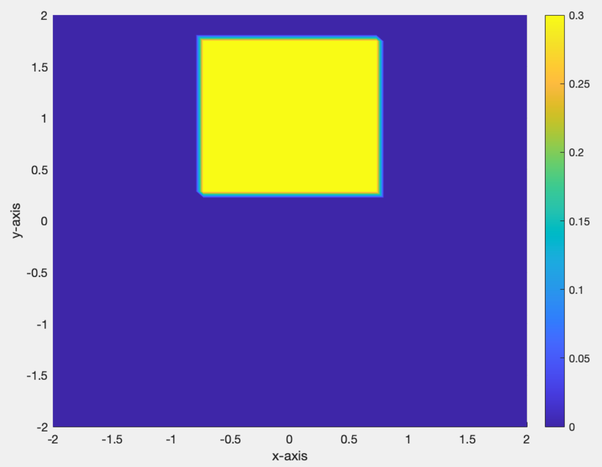

Figure 3: True profile

Example 1: Square profile, $\eta = 0.3$

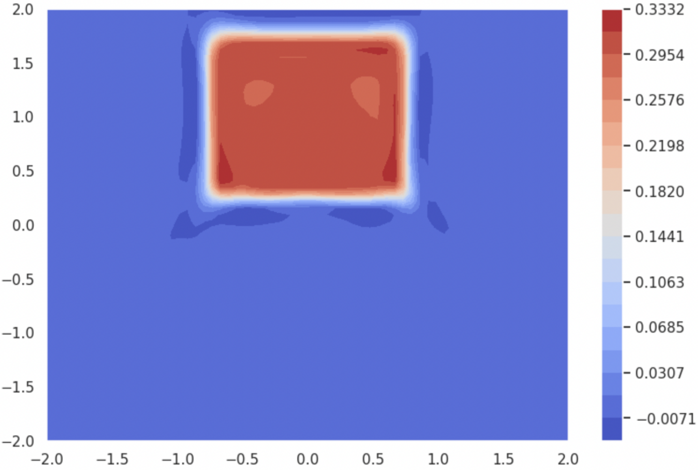

Figure 4: Prediction with 0% synthetic noise

Example 1: Square profile, $\eta = 0.3$

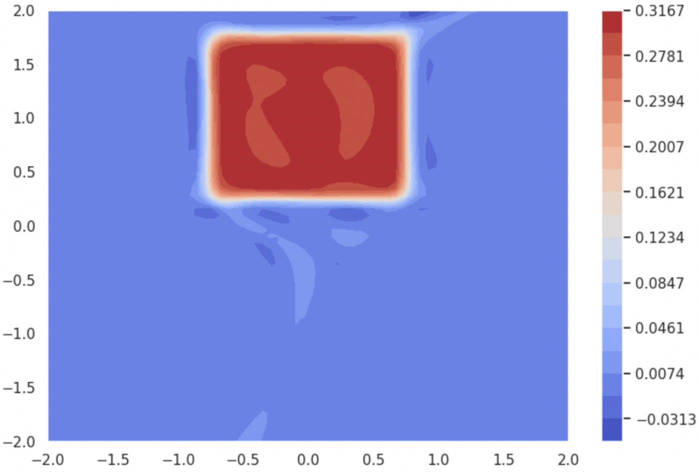

Figure 5: Prediction with 10% synthetic noise

Example 1: Square profile, $\eta = 0.3$

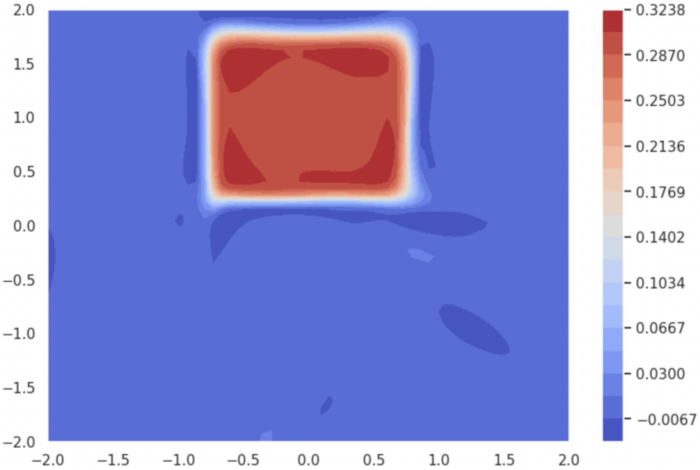

Figure 6: Prediction with 20% synthetic noise

Example 2: Elliptical profile, $\eta = 3.0$

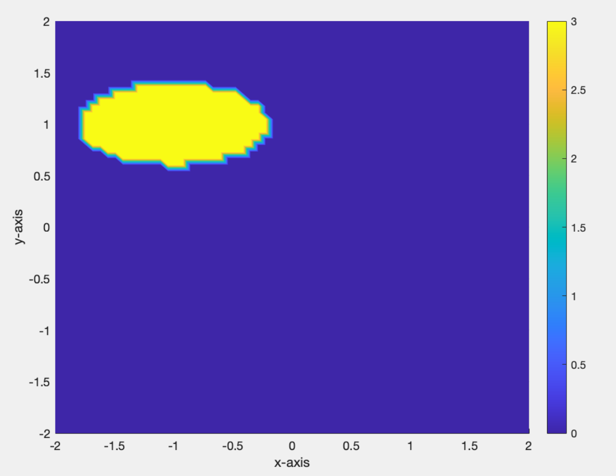

Figure 7: True profile

Example 2: Elliptical profile, $\eta = 3.0$

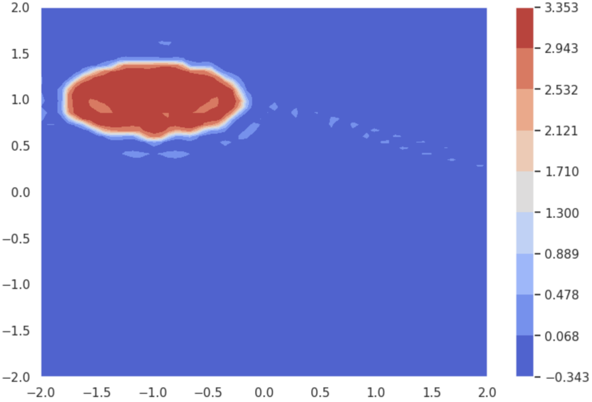

Figure 8: Prediction with 0% synthetic noise

Example 2: Elliptical profile, $\eta = 3.0$

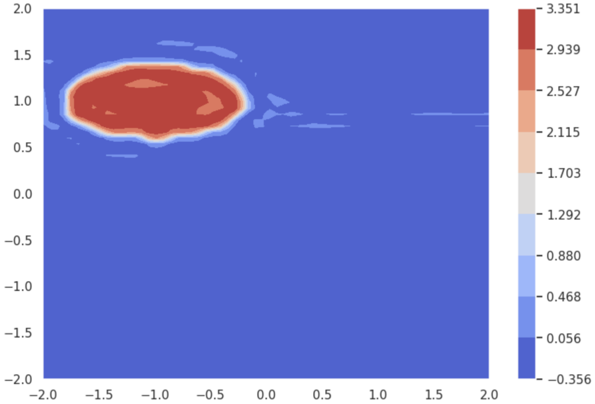

Figure 9: Prediction with 10% synthetic noise

Example 2: Elliptical profile, $\eta = 3.0$

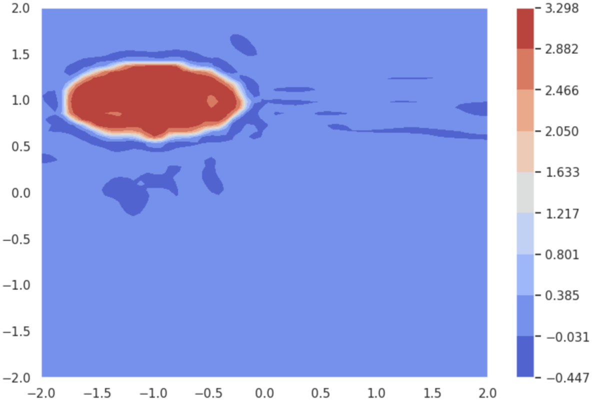

Figure 10: Prediction with 20% synthetic noise

Example 3: Austria profile, $\eta = 2.0$

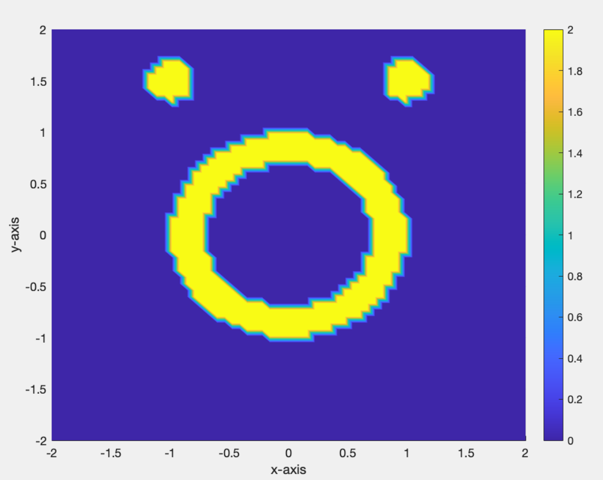

Figure 11: True profile

Example 3: Austria profile, $\eta = 2.0$

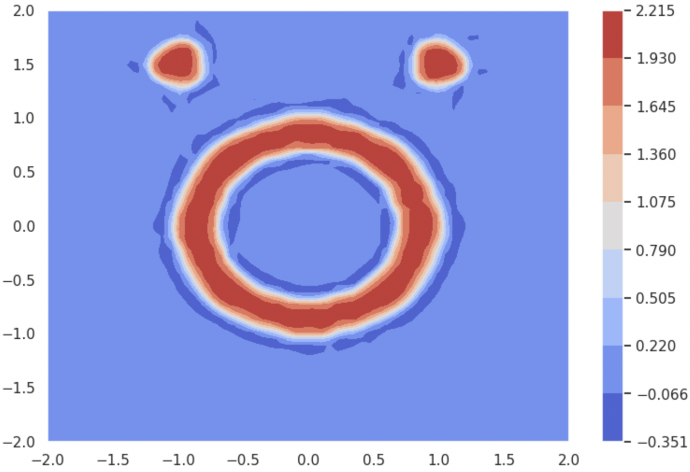

Figure 12: Prediction with 0% synthetic noise

Example 3: Austria profile, $\eta = 2.0$

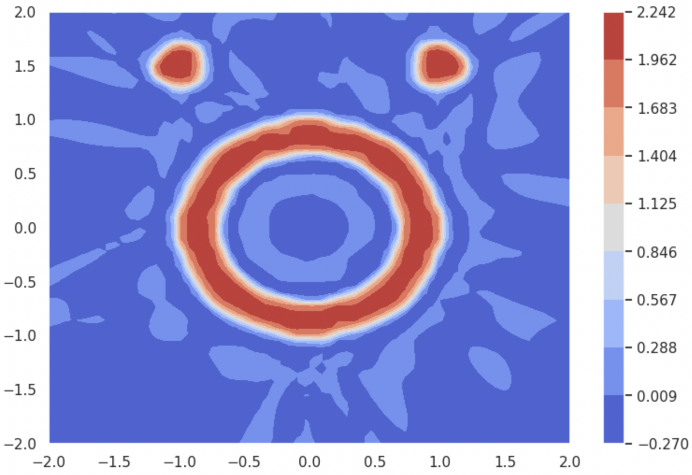

Figure 13: Prediction with 10% synthetic noise

Example 3: Austria profile, $\eta = 2.0$

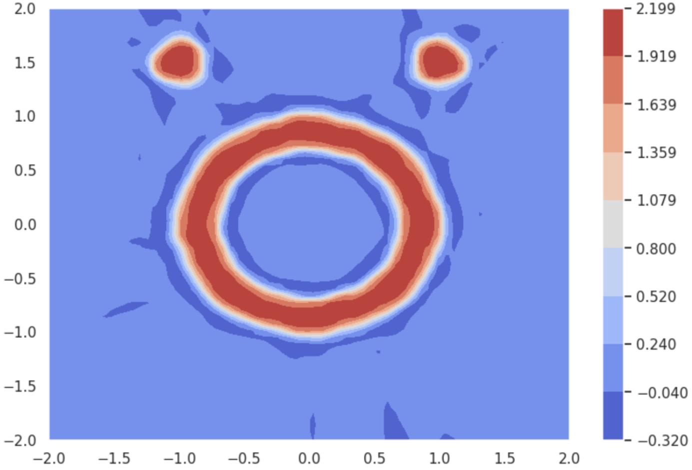

Figure 14: Prediction with 20% synthetic noise

Conclusion

We have developed an unsupervised and model-informed deep learning algorithm to solve inverse scattering problems

- The algorithm first extracts geometrical information about the target from boundary data using an imaging function.
- The algorithm incorporates a simple model integral equation into the deep learning process instead of PDE models, eliminating the need for automatic differentiation.
- The method is highly robust against noisy data, ensuring stability and accuracy in practical applications.

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Thank you for listening!