Numerical Methods for Inverse Source Problems

Aravinth Krishnan Ravi

Kansas State University

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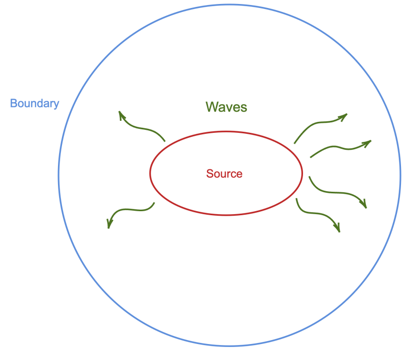
Committee Members: Dinh-Liem Nguyen (Advisor), Nathan Albin, Anna Y. Zemlyanova, William Hsu

Outline

Outline

Direct Source Problem



Direct Source Problem: Given a source that radiates waves, determine the waves.

Mathematical Formulation of Direct Source Problem

- Let $f \in L^2(\mathbb{R}^2)$ be a bounded function with compact support.
- The direct source problem:

$$\begin{cases} \Delta u + k^2 u = f, & \text{in } \mathbb{R}^2 \\ \frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r^2}\right), & \text{as } r = |x| \to \infty. \end{cases}$$

Given f, determine u.

Green's Function

Definition 1 (Green's Function)

The Green's function $\Phi(x, y)$, is the fundamental solution to the Helmholtz equation and is defined as follows:

$$\Phi(x, y) = \frac{i}{4} H_0^1(k|x - y|),$$

where $x \neq y$.

Here, $H_0^1(k|x - y|)$ is called the Hankel function of the first kind and of order 0. It is defined as follows:

$$H_0^1(z) = J_0(z) + iY_0(z),$$

where J_0 and Y_0 are the Bessel function of the first kind and second kind respectively. Important Facts:

- $\Phi(x, y)$ satisfies the Sommerfeld radiation condition.
- $\Phi(x, y)$ has a singularity at $x = y$.

Green's Representation Theorem

Theorem 2

Let $u \in H^2(\Omega)$. Then, we have

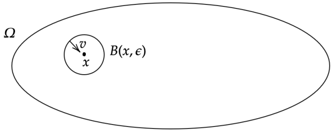
$$u(x) = \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} \right\} dS(y)$$
$$- \int_{\Omega} \left\{ \Delta u(y) + k^2 u \right\} \Phi(x,y) dy, x \in \Omega,$$

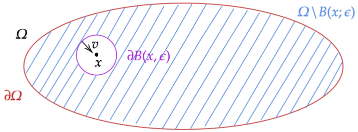
In particular, if $\Delta u + k^2 u = 0$ in Ω, then

$$u(x) = \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} \right\} dS(y), \ x \in \Omega.$$

We will first show that if $u \in C^2(\overline{\Omega})$, then the theorem holds. Then, we can use density arguments to establish the results when $u \in H^2(\Omega)$.

Singularity at $x = y$ prevents us from using Φ in Green's second identity. Thus, we circumscribe an arbitrary fixed point $x \in \Omega$ with a ball $B(x;\epsilon)$ contained in Ω.





Then, for sufficiently small $\epsilon > 0$, applying Green's second identity to u and ϕ, we have:

$$\int_{\Omega\setminus B(x,\epsilon)} \Delta u(y)\Phi(x,y) - u(y)\Delta\Phi(x,y)dy = \int_{\partial\Omega} \frac{\partial u(y)}{\partial n}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)}dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial u}{\partial n}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)}dS(y).$$

Main Idea

Using the definition of the Green's function and arguing by estimates, we obtain:

$$\int_{\partial B(x,\epsilon)} \frac{\partial u}{\partial n}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)}\,dS(y) = \underbrace{\int_{\partial B(x,\epsilon)} \frac{\partial u}{\partial n}\Phi(x,y)dS(y)}_{\to 0} - \underbrace{\int_{\partial B(x,\epsilon)} u(y)\frac{\partial \Phi(x,y)}{\partial n(y)}\,dS(y)}_{\to u(x)},$$

as $\epsilon \to 0$.

On the other hand, from the fact that $\Delta\Phi(x,y) + k^2\Phi(x,y) = 0$, we get

$$\int_{\Omega\setminus B(x,\epsilon)} \Delta u(y)\Phi(x,y) - u(y)\Delta\Phi(x,y)dy = \int_{\Omega\setminus B(x,\epsilon)} \left(\Delta u(y) + k^2 u(y)\right)\Phi(x,y)dy.$$

Hence, as $\epsilon \to 0$, we get:

$$\int_{\Omega} \left(\Delta u(y) + k^2 u(y)\right)\Phi(x,y)dy = \int_{\partial\Omega} \frac{\partial u(y)}{\partial n}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)}\,dS(y) - u(x),$$

which is our desired result.

Sobolev Case

Since $C^2(\overline{\Omega})$ is dense in $H^2(\Omega)$, for $u \in H^2(\Omega)$, there exist $(u_m) \subset C^2(\overline{\Omega})$ such that $u_m \to u$ in $H^2(\Omega)$. Then, by trace theorem and embedding arguments,

$$||u_m - u||_{L^2(\partial\Omega)} \leq C||u_m - u||_{H^1(\Omega)} \leq C||u_m - u||_{H^2(\Omega)}$$
$$\left|\left|\frac{\partial u_m}{\partial n} - \frac{\partial u}{\partial n}\right|\right|_{L^2(\partial\Omega)} \leq C\left|\left|\frac{\partial u_m}{\partial n} - \frac{\partial u}{\partial n}\right|\right|_{H^1(\Omega)} \leq C||u_m - u||_{H^2(\Omega)},$$

it follows that:

$$||\Delta u_m - \Delta u||_{L^2(\partial\Omega)} \leq C||u_m - u||_{H^2(\Omega)}.$$

So, Green's representation theorem also holds for $u \in H^2(\Omega)$.

Exterior Green's Representation

Theorem 3

Let $u \in H^2_{loc}(\mathbb{R}^2 \setminus \overline{\Omega})$. If u is a radiating solution, then

$$u(x) = \int_{\partial\Omega} u(y)\frac{\partial\Phi(x,y)}{\partial n(y)} - \frac{\partial u(y)}{\partial n}\Phi(x,y)dS(y), x \in \mathbb{R}^2 \setminus \overline{\Omega}.$$

By the Interior Green Representation Theorem in Ω_r, $\forall x \in \mathbb{R}^2 \setminus \overline{\Omega}$,

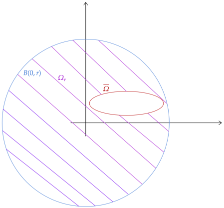
$$\begin{aligned} u(x) = &- \int_{\partial\Omega} \frac{\partial u(y)}{\partial n}\Phi(x,y) - u(y)\frac{\Phi(x,y)}{\partial n(y)}dS(y) \\ &- \int_{\partial B(0,r)} \frac{\partial u(y)}{\partial n}\Phi(x,y) - u(y)\frac{\Phi(x,y)}{\partial n(y)}dS(y). \end{aligned}$$

Claim: As $r \to \infty$,

$$\int_{\partial B(0,r)} \frac{\partial u(y)}{\partial n}\Phi(x,y) - u(y)\frac{\Phi(x,y)}{\partial n(y)}dS(y) \to 0$$

Main Idea

Step 1: Show $\int_{B(0,r)} |u|^2 ds = \mathcal{O}(1)$.



From the radiation condition, it follows that:

$$\int_{\partial B(0,r)} \left|\frac{\partial u}{\partial r} - iku\right|^2 dS(y) = \int_{\partial B(0,r)} \left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2 + 2kIm\left(u\frac{\partial \overline{u}}{\partial r}\right) dS(y) \to 0$$

as $r \to \infty$.

Main Idea

Apply Green's first theorem to Ω_r:

$$\int_{B(0,r)} u\frac{\partial \overline{u}}{\partial r}ds = \int_{\partial\Omega} u\frac{\partial \overline{u}}{\partial r}ds - k^2\int_{\Omega_r}|u|^2dy + \int_{\Omega_r}|\nabla u|^2dy.$$

Take imaginary part of both sides, apply it to the previous equation, we get:

$$\lim_{r\to\infty}\int_{B(0,r)}\left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2ds = -2Im\int_{\partial\Omega} u\frac{\partial \overline{u}}{\partial r}ds.$$

Both terms on the LHS are non-negative. RHS is indepedent of r. Thus, both terms on the LHS are individually bounded as $r \to \infty$, proving our claim.

Main Idea

Step 2: By the Cauchy-Schwarz inequality, the claim above and the fact that Φ satisfies the radiation condition,

$$\underbrace{\int_{\partial B(0,r)} u(y)\left(\frac{\partial \Phi(x,y)}{\partial n} - ik\Phi(x,y)\right) dS(y)}_{I_1} \to 0 \text{ as } r \to \infty.$$

By Cauchy-Schwarz inequality, radiation condition for u, and $\Phi(x,y) = \mathcal{O}(\frac{1}{r})$,

$$\underbrace{\int_{\partial B(0,r)} \Phi(x,y)\left(\frac{\partial u(y)}{\partial n} - iku(y)\right) dS(y)}_{I_2} \to 0 \text{ as } r \to \infty.$$

And from there, we obtain

$$I_1 - I_2 = \int_{\partial B(0,r)} u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} - \frac{\partial u(y)}{\partial n}\Phi(x,y) dS(y) \to 0 \text{ as } r \to \infty.$$

Corollary

Theorem 4

If u satisfies

$$\Delta u + k^2 u = 0, \ \textit{in } \mathbb{R}^2$$
$$\frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r^2}\right), r = |x| \to \infty,$$

then $u = 0$ *in* $\mathbb{R}^2$.

Cases:

- $\mathbb{R}^2 \setminus \overline{\Omega}$
- $\overline{\Omega}$

Main Idea: $\mathbb{R}^2 \setminus \overline{\Omega}$

For $x \in \mathbb{R}^2 \setminus \overline{\Omega}$, by the exterior Green representation theorem:

$$u(x) = \int_{\partial\Omega} u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} - \frac{\partial u(y)}{\partial n}\Phi(x,y)ds(y).$$

By the Green's second identity,

$$\int_{\partial\Omega} u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} - \frac{\partial u(y)}{n(y)}\Phi(x,y)dy$$
$$= \int_{\Omega} u(y)\Delta\Phi(x,y) - \Delta u(y)\Phi(x,y)dy$$

Since $\Delta u + k^2 u = 0,$ in $\mathbb{R}^2$, we have:

$$u(x) = \int_{\Omega} \left(\Delta\Phi(x,y) + k^2\Phi(x,y)\right) u(y)dy = 0. \qquad (1)$$

Main Idea: $\overline{\Omega}$

For $x \in \Omega$, by the Interior Green's Representation:

$$u(x) = \int_{\partial\Omega} \left(\frac{\partial u(y)}{\partial n} \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \right) ds(y).$$

Since u is analytic in $\mathbb{R}^2 \implies u$ is continuous across the boundary.

Therefore, $u(y) = 0$ on $\partial\Omega$ and hence, $u(x) = 0, \text{ for } x \in \Omega$.

Together with (1), it follows that $u = 0 \in \mathbb{R}^2$.

Existence and Uniqueness of Solution

Theorem 5

The problem

$$\Delta u + k^2 u = f, f \in C_0^\infty(\Omega)$$
$$\frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r^2}\right), r = |x| \to \infty$$

has a unique solution.

Assume that u_1 and u_2 are the solutions to the PDE Problem. Then,

$$\Delta(u_1 - u_2) + k^2(u_1 - u_2) = 0,$$
$$\frac{\partial(u_1 - u_2)}{\partial r} - ik(u_1 - u_2) = \mathcal{O}(\frac{1}{r^2}),$$

as $r = |x| \to \infty$. Then, by theorem 4, we have $u_1 - u_2 = 0$ in $\mathbb{R}^2$.

Main Idea

Let $u(x) = \Phi * f = \int_{\mathbb{R}^2} \Phi(x,y) f(y) dy$. First, it is clear that $\Delta u + k^2 u = f$ in the distributional sense.

Next, note that u can be expressed as follows, using the far field representation of Φ:

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^2} \Phi(x,y) f(y) dy \\ &= \int_{\mathbb{R}^2} \Phi(x) \left(\Phi^{\infty}(\hat{x}) + \mathcal{O}\left(\frac{1}{r}\right)\right) f(y) dy \\ &= \Phi(x) \int_{\mathbb{R}^2} \Phi^{\infty}(\hat{x}) f(y) + \mathcal{O}\left(\frac{1}{r}\right) f(y) dy. \end{aligned}$$

The form above establishes that u behaves like Φ as $|x| \to \infty$. Since, Φ satsifies the Sommerfeld radiation condition, it follows that u satisfies the radiation condition.

Main Idea

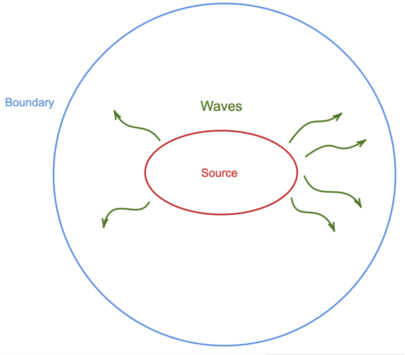
For the case, $f \in L^2(\Omega)$, we can use density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$ and get the same result.

Definition 6 (Volume Potential)

$u(x) = \int_{\mathbb{R}^2} \Phi(x, y) f(y) dy$ is called the volume potential.

Outline

Inverse Source Problem



Direct Source Problem: Given a source that radiates waves, determine the waves.

Inverse source problem: Given information of the waves at the boundary, recover the physical and geometrical information about the source.

Mathematical formulation of inverse source problem

- Let $f \in L^2(\Omega)$, where Ω is a large disk such that $supp(f) = D \subset \Omega$.

$$\begin{cases} \Delta u + k^2 u = f, & \text{in } \mathbb{R}^2 \\ \frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r^2}\right), & \text{as } r = |x| \to \infty. \end{cases}$$

- Inverse problem: **Given** $u(\cdot, k)|_{\partial\Omega}$ for $k \in [k_1, k_2]$, **determine** f.

Uniqueness and stability of solution for the inverse source problem:

- Bao, G., Lin, J., & Triki, F. (2010). A multi-frequency inverse source problem. Journal of Differential Equations, 249, 3443-3465.

Theorem 7

Given a set of real numbers $[k_1, k_2]$, the measurements $u(\cdot, k)|_{\partial\Omega}$, where $k \in [k_1, k_2]$, determine uniquely the source function f.

Numerical methods to solve inverse source problems

Traditional numerical schemes (far-from-complete list):

- *Bao, G., Lin, J., & Triki, F. (2011). Numerical solution of the inverse source problem for the Helmholtz Equation with multiple frequency data.*
- *Acosta, S., Chow, S., Taylor, J., & Villamizar, V. (2012). On the multi-frequency inverse source problem in heterogeneous media. Inverse Problems, 28.*
- *Eller, M., & Valdivia, N. (2009). Acoustic source identification using multiple frequency information. Inverse Problems, 25, 115005.*
- Bao, G., Lin, J., & Triki, F. (2010). A multi-frequency inverse source problem. Journal of Differential Equations, 249, 3443-3465.
- Kress, R., & Rundell, W. (2013). Reconstruction of extended sources for the Helmholtz equation. Inverse Problems, 29.
- Karamehmedovi'c, M., Kirkeby, A., & Knudsen, K. (2018). Stable source reconstruction from a finite number of measurements in the multi-frequency inverse source problem. Inverse Problems, 34.
- Eibert, T.F. (2023). Multiple-Frequency Preconditioned Iterative Inverse Source Solutions. IEEE Transactions on Microwave Theory and Techniques, 71, 2842-2853.

Severely illposed $\implies$ numerical schemes require regularization strategies

Deep learning based algorithms for multifrequency inverse source problems

Deep learning-based algorithms (far-from-complete list):

- *Dong, Y., Sadiq, K., Scherzer, O., & Schotland, J.C. (2024). Computational inverse scattering with internal sources: a reproducing kernel Hilbert space approach. Physical review. E, 110 6-2, 065302 .*
- *Meng, S., & Zhang, B. (2024). A Kernel Machine Learning for Inverse Source and Scattering Problems. SIAM J. Numer. Anal., 62, 1443-1464.*
- *Zhang, H., & Liu, J. (2023). Solving an inverse source problem by deep neural network method with convergence and error analysis. Inverse Problems, 39.*

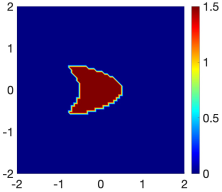
Solve inverse source problems with "internal" data!

Outline

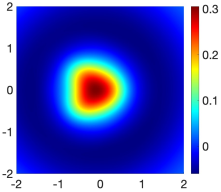
Proposed Algorithm

We developed a 2-step **unsupervised** deep learning algorithm to solve the inverse source problem:

- Step 1: Convert the boundary data into an imaging function $\mathcal{I}(z)$ which encodes geometrical information about f.
- Step 2: Use $\mathcal{I}(z)$ as input for training. A model equation for $\mathcal{I}(z)$ and f is employed to train a neural network which predicts f.



(a) Original source



(b) Data from $\mathcal{I}$

Stable Imaging Function

- For each $k \in [k_1, k_2]$, for $z \in \mathbb{R}^2$, define the function

$$\mathcal{I}(z,k) := \int_{\partial\Omega} u(x,k)\overline{\Phi_k(x,z)}\, dS(x),$$

where $\Phi_k(x,z)$ denotes the Green's function.

We assume the noisy data u_δ satisfies

$$\|u - u_\delta\|_{L^2(\partial\Omega\times\mathbb{S})} \leq \delta\|u\|_{L^2(\partial\Omega\times\mathbb{S})}.$$

Theorem 8

Let $\mathcal{I}_\delta(z)$ the imaging function with noisy data u_δ. Then

$$|\mathcal{I}(z) - \mathcal{I}_\delta(z)| \leq C\delta, \quad \text{for all } z \in \mathbb{R}^2,$$

where C is a positive constant which is independent of z and δ.

Main Idea

$$
\begin{aligned}
|\mathcal{I}(z) - \mathcal{I}_\delta(z)| &= \left| \int_{\partial\Omega} u(x,k)\overline{\Phi_k(x,z)}\, dS(x) - \int_{\partial\Omega} u_\delta(x,k)\overline{\Phi_k(x,z)}\, dS(x) \right| \\
&= \left| \int_{\partial\Omega} (u(x,k) - u_\delta(x,k))\overline{\Phi_k(x,z)}\, dS(x) \right| \\
&\leq \int_{\partial\Omega} ||u(x,k) - u_\delta(x,k)|| \left|\left|\overline{\Phi_k(x,z)}\right|\right| dS(x) \\
&\leq \int_{\partial\Omega} \delta\, ||u(x,k)|| \left|\left|\overline{\Phi_k(x,z)}\right|\right| dS(x) \\
&= \delta \underbrace{\int_{\partial\Omega} ||u(x,k)|| \left|\left|\overline{\Phi_k(x,z)}\right|\right| dS(x)}_{C}
\end{aligned}
$$

Helmholtz Kirchhoff identity

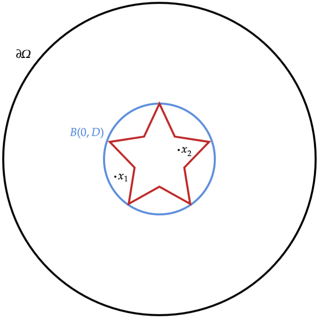
Theorem 9 (Helmholtz Kirchhoff identity)

Let $supp(f) \subset B(0,D) \subset \partial\Omega$. *Then* $\forall x_1, x_2 \in B(0,D)$, *we have*

$$\Phi(x_1,x_2) - \overline{\Phi(x_1,x_2)} = 2ki \int_{\partial\Omega} \overline{\Phi(x_1,y)}\Phi(x_2,y)dS(y).$$

Key Idea behind Proof:

- Green's Identity
- Sommerfeld radiation condition



Significance of the model equation

Theorem 10

The imaging function $\mathcal{I}(z)$ satisfies

$$\mathcal{I}(z,k) = \frac{1}{4k}\int_\Omega J_0(k|y-z|)\, f(y)\, dy$$

Proof: Using the volume potential and the Helmholtz-Kirchhoff identity.

Significance:

- A simple and nice connection between $\mathcal{I}(z)$ and f.
- This equation will be used as a model equation in training our neural network with data $\mathcal{I}(z)$.

Main idea

For each $k \in [k_1, k_2]$, for $z \in \mathbb{R}^2$,

$$\begin{aligned}\int_{\partial\Omega} u(x)\overline{\Phi(x,z)}\,dS(x) &= \int_{\partial\Omega}\int_{\mathbb{R}^2} \Phi(x,y)f(y)dy\overline{\Phi(x,z)}\,dS(x) \\ &= \int_{\Omega}\int_{\partial\Omega} \Phi(x,y)\overline{\Phi(x,z)}\,dS(x)f(y)\,dy \\ &= \frac{1}{4k}\int_{\Omega} J_0(k|y-z|)f(y)\,dy\end{aligned}$$

Idea behind PINN

- Concept leverages on the fact that deep neural networks are universal approximators
- Based on the elementary observation that: If a surrogate function $w_\theta \in \mathcal{H}$ satisfies

$$\mathcal{I}(z,k) - \frac{1}{4k}\int_\Omega J_0(k|y-z|)\, w_\theta(y)\, dy \approx 0,$$

then $f \approx w_\theta$.

Training Process

Ansatz Space/Hypothesis space $\mathcal{H}$: Space of Neural Networks with the following configuration:

- Depth of Neural Network: 4
- Width of Neural Network: 64
- Activation Functions: Hyperbolic Tangent Function

Training Process

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With the chosen space $\mathcal{H}$ in place, our physics-based loss functional $\mathcal{L}[\cdot]$ is:

$$\mathcal{L} : \mathcal{H} \to \mathbb{R}$$

$$f_\theta \mapsto \frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{N}\sum_{j=1}^{N}\left[\mathcal{I}(z_j, k_i) - \frac{1}{4k_i}\int_\Omega J_0(k_i|z_j - y|)f_\theta(y)dy\right]\right).$$

Training Process

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Minimization Problem:

$$\min_{w_\theta \in \mathcal{H}} \frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{N}\sum_{j=1}^{N}\left[\mathcal{I}(z_j, k_i) - \frac{1}{4k_i}\int_\Omega J_0(k_i|z_j - y|) w_\theta(y)dy\right]\right)$$

Let $z = (x, y)^\top \in \mathbb{R}^2$.

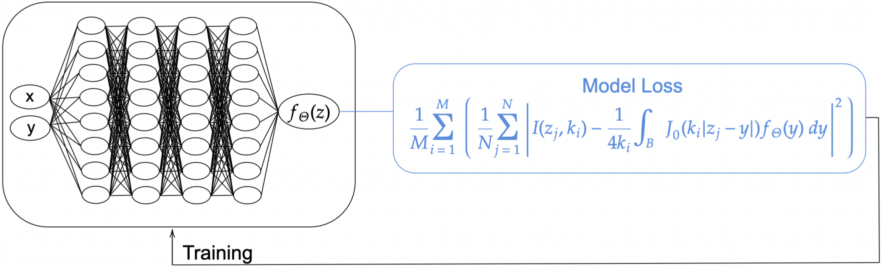
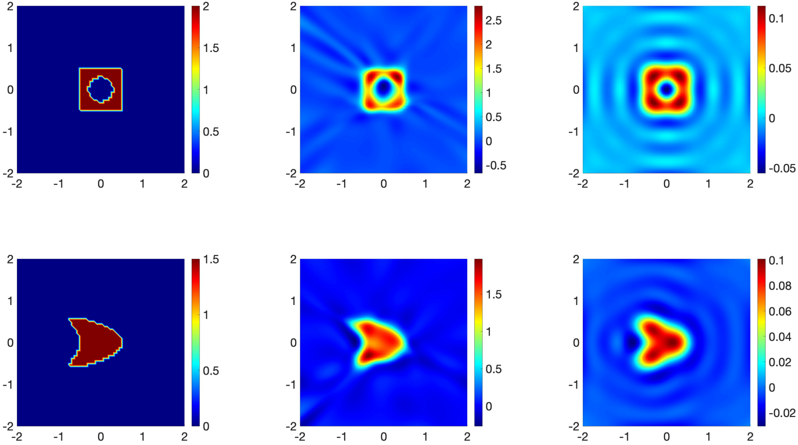


Figure 3: Visual representation of the model-informed neural network

Training is executed by:

- Adams Optimizer
- Glorot Uniform Weight Initialiser

Reconstruction with 20% artifical noise introduced to the boundary data:



(a) Original Source (b) Predicted (c) Least Square

Review

We have developed an unsupervised, model-informed deep learning algorithm to solve inverse source problems

- In step 1, regularization is done automatically $\implies$ no need for additional regularization
- The algorithm incorporates a simple model integral equation into the deep learning process instead of PDE models, eliminating the need for automatic differentiation $\implies$ computationally cheaper
- The method is highly robust against noisy data, ensuring stability and accuracy in practical applications.

Outline

Extension - Inverse Scattering Problems

Can be extended to solve inverse scattering problems too

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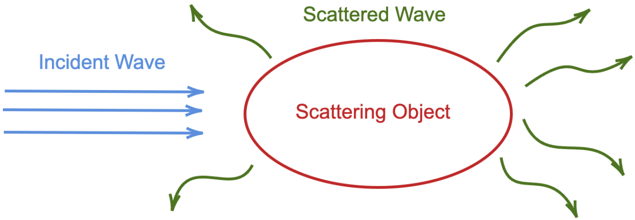


Inverse scattering problem: Determine the *scattering object* from boundary measurements of scattered wave (for several incident waves).

Applications: Radar, non-destructive testing, geophysical exploration, medical imaging, ...

Mathematical Formulation

- Let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be bounded function satisfying $\eta = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Let Ω be large disk such that $D \subset \Omega$. Consider incident waves

$$u^{in}(x, d) = e^{ikd\cdot x}$$

where $d \in \mathbb{S}^1 := \{x \in \mathbb{R}^2 : |x| = 1\}$.

- Consider the following model problem

$$\begin{cases} \Delta u + k^2(1 + \eta(x))u = 0, & \text{in } \mathbb{R}^2 \\ u = u^{in} + u^{sc}, & \\ \lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^{sc}}{\partial r} - iku^{sc}\right) = 0, & r = |x|. \end{cases}$$

- **Inverse Problem:** **Given** $u^{sc}(\cdot, d)|_{\partial\Omega}$ for all $d \in \mathbb{S}^1$, **determine** η.

Deep Learning for inverse scattering

Supervised learning-based algorithms (far-from-complete list)

- *Wei, Z., & Chen, X. (2019). Deep-Learning Schemes for Full-Wave Nonlinear Inverse Scattering Problems. IEEE Transactions on Geoscience and Remote Sensing, 57, 1849-1860.*
- *Khoo, Y., & Ying, L. (2018). SwitchNet: a neural network model for forward and inverse scattering problems. ArXiv, abs/1810.09675.*
- *Sanghvi, Y., Kalepu, Y., & Khankhoje, U.K. (2020). Embedding Deep Learning in Inverse Scattering Problems. IEEE Transactions on Computational Imaging, 6, 46-56.*
- *Chen, X., Wei, Z., Li, M., & Rocca, P. (2020). A Review of Deep Learning Approaches for Inverse Scattering Problems (Invited Review). Progress In Electromagnetics Research.*
- *Gao, Y., Liu, H., Wang, X., & Zhang, K. (2021). On an artificial neural network for inverse scattering problems. J. Comput. Phys., 448, 110771.*
- *Zhou, M., Han, J., Rachh, M., Borges, C. (2022). A Neural Network Warm-Start Approach for the Inverse Acoustic Obstacle Scattering Problem. ArXiv, abs/2212.08736.*
- ...

Reconstruct unknown scattering objects that are similar or closely related to known training datasets!

Deep Learning for inverse scattering

Unsupervised learning: Physics Informed Neural Networks (PINNs)

- *Raissi, M., Perdikaris, P., & Karniadakis, G.E. (2019). Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. J. Comput. Phys., 378, 686-707.*
- *Chen, Y., Lu, L., Karniadakis, G.E., & Dal Negro, L. (2019). Physics-informed neural networks for inverse problems in nano-optics and metamaterials. Optics express, 28 8, 11618-11633 .*
- *Chen, Y., & Dal Negro, L. (2021). Physics-informed neural networks for imaging and parameter retrieval of photonic nanostructures from near-field data. APL Photonics.*
- *Pokkunuru, A., Rooshenas, P., Strauss, T., Abhishek, A., & Khan, T.R. (2023). Improved Training of Physics-Informed Neural Networks Using Energy-Based Priors: a Study on Electrical Impedance Tomography. International Conference on Learning Representations.*

Study inverse design problems with “internal” data!

Imaging Function

- For $z \in \mathbb{R}^2$, define the function

$$\mathcal{I}(z) := \int_S \int_{\partial\Omega} u^{sc}(x, d)\overline{\Phi(x, z)}\, ds(x)\, \Phi^\infty(d, z)\, ds(d),$$

  where $\Phi(x, z)$ and $\Phi^\infty(d, z)$ denote the Green's function and its scattering amplitude respectively.

With the Helmholtz-Kirchhoff Identity, Funk-Hecke Formula and Born approximation, we can derive

$$\mathcal{I}(z) - \frac{k\pi}{2} \int_D J_0^2(k|y - z|)\, \eta(y)\, dy = 0.$$

- A simple and nice connection between $\mathcal{I}(z)$ and η.
- This equation will be used as a (simplified) model equation in training our neural network with data $\mathcal{I}(z)$.

Stable Imaging Function

We assume the noisy data u_δ^{sc} satisfies

$$\|u^{sc} - u_\delta^{sc}\|_{L^2(\partial\Omega\times\mathbb{S})} \leq \delta\|u^{sc}\|_{L^2(\partial\Omega\times\mathbb{S})}.$$

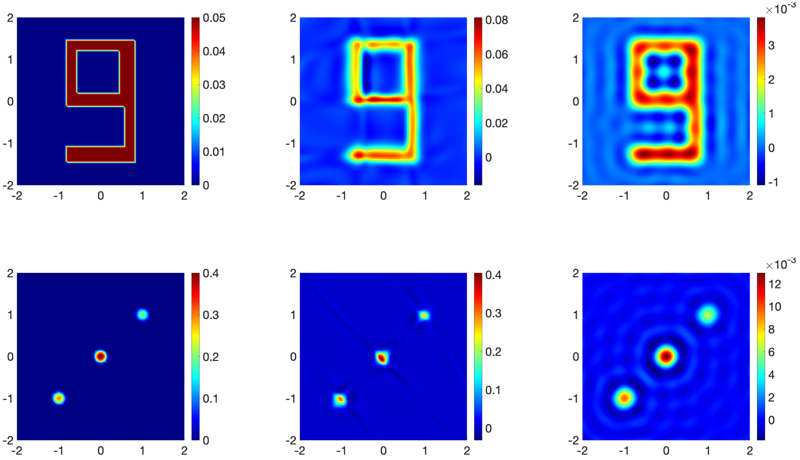
Theorem 11

Let $I_\delta(z)$ the imaging function with noisy data u_δ^{sc}. Then

$$|\mathcal{I}(z) - \mathcal{I}_\delta(z)| \leq C\delta, \quad \text{for all } z \in \mathbb{R}^2,$$

where C is a positive constant which is independent of z and δ.

Some numerical results



(a) Original Scatterer (b) Predicted (c) Least Square

Review

We have developed an unsupervised, model-informed deep learning algorithm to solve inverse scattering problems

- Unsupervised $\implies$ greater generalisability
- Similar to the case of the algorithm for the inverse source problem, the algorithm is computationally cheap
- Again, the method is highly robust against noisy data, ensuring stability and accuracy in practical applications.

Thank you for listening!