# Tropical Geometry

## Tropical Geometry

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# A Brief Recap on Tropical Algebra

- Set of Tropical Numbers, $\mathbb{T} := \mathbb{R} \cup \{\infty\}$
- Operations on $\mathbb{T}$
  - Tropical Addition: $"x + y" := min\{x, y\}$
  - Tropical Multiplication: $"x \cdot y" := x + y$
- With our usual conventions, $(\infty)$ is our additive identity.
  - $\forall x \in \mathbb{T}, "x + (\infty)" = min\{x, \infty\} = x$, and
  - $\forall x \in \mathbb{T}, "x \cdot (\infty)" = x + (\infty) = (\infty)$
- So, we can see that $(\mathbb{T}, " + ", " \cdot ")$ satisfies all the field axioms except the existence of tropical additive inverse. So, $(\mathbb{T}, " + ", " \cdot ")$ forms a semi-field.
- Semi-rings vs semi-fields

# Tropical Polynomials: 1 Dimensional Case

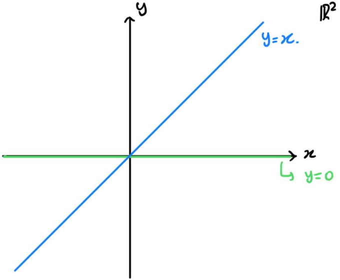
- A Tropical Polynomial $P(x) = \text{"}\sum_{i=0}^{d} a_i x^i\text{"}$ is viewed as

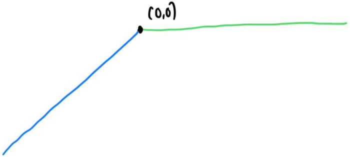
$$\text{"}\sum_{i=0}^{d} a_i x^i\text{"} = min_{i=1}^{d}\{a_i + ix\}$$

- So, a tropical polynomial is a convex piecewise affine function and each piece has as integer slope
- Roots of a tropical polynomial: All the points $x_0 \in \mathbb{T}$ for which the graph of $P(x)$ has a corner (bends) at $x_0$. This is equivalent to $P(x_0)$ being equal to the value of at least 2 of its monomials evaluated at $x_0$.

# Example 1

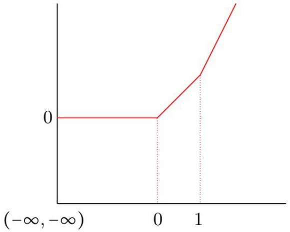
- Consider the function $f = "0 + x" := min\{0, x\}$



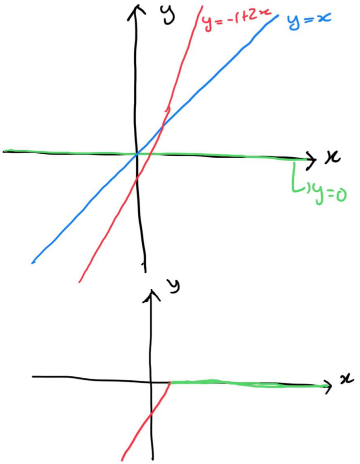



# Example 2

- Consider the equation
  $P(x) = "0 + x + (-1)x^{2}" := min\{0, x, -1 + 2x\}$



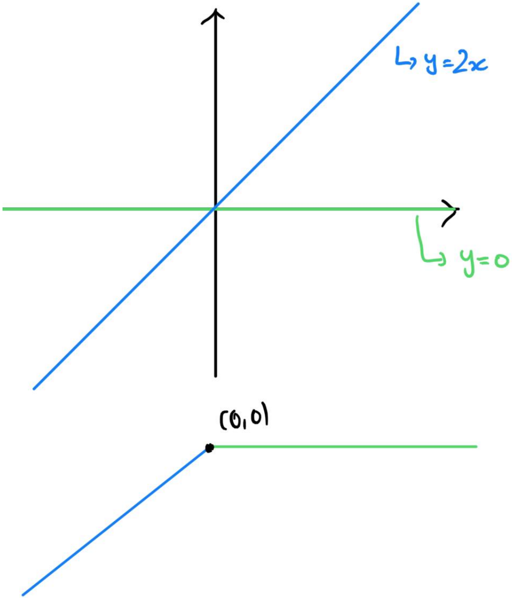

b) $P(x) =$ "$0 + x + (-1)x^{2}$"




- Max vs Min??

# Example 3

- Consider the function $f = "0 + x^2"$



# Tropical Polynomial in 2 variables

- A Tropical Polynomial in 2 variables is $P(x,y) = " \sum_{(i,j)\in A} a_{(i,j)} x^i y^j " = min_{(i,j)\in A}(a_{(i,j)} + ix + jy)$, where $A$ is a finite subset of $(\mathbb{Z}_{\geq 0})^2$.
- Thus, a tropical polynomial in 2 dimensions is a convex piecewise affine function
- The roots of the tropical polynomial is the corner locus of this function. That is to say,

$$\tilde{V}(P) = \{(x_0, y_0) \in \mathbb{R}^2 : (i,j) \neq (k,l), P(x_0,y_0) = " a_{i,j} x_0^i y_0^j " \\ = " a_{k,l} x_0^k y_0^l " \}$$

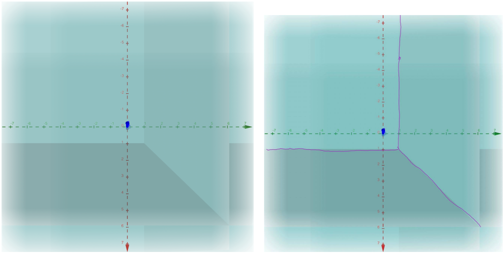
- Said another way, a tropical curve $C$ consists of all points $(x_0, y_0) \in \mathbb{T}^2$ for which the minimum of $P(x,y)$ is obtained at least twice at $(x_0, y_0)$

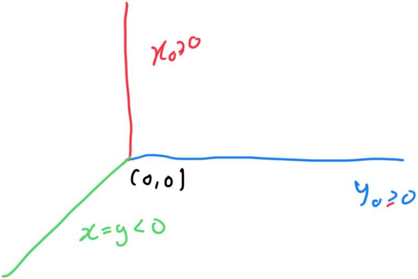
# Example 1

- Consider the equation $f = "0 + x + y" := min\{0, x, y\}$
- We must find points $(x_0, y_0) \in \mathbb{R}^2$ that satisfy one of the following 3 conditions:
  - $x_0 = 0 \leq y_0$
  - $y_0 = 0 \leq x_0$
  - $x_0 = y_0 \leq 0$
- Then, the set $\tilde{V}(P)$ is made of three standard half-lines:
  - $\{(0, y) \in \mathbb{R}^2 | y \geq 0\}$
  - $\{(x, 0) \in \mathbb{R}^2 | x \geq 0\}$
  - $\{(x, x) \in \mathbb{R}^2 | x \leq 0\}$
- Then, the set $\tilde{V}(P)$ is a piecewise linear graph in $\mathbb{R}^2$.

# Example Continued

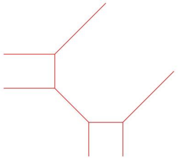
- we can visualise this in $\mathbb{R}^3$.



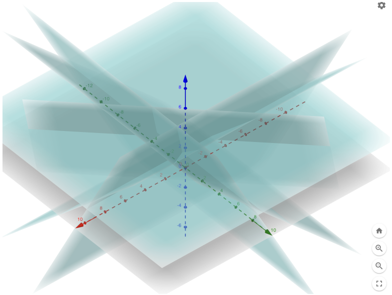


# Example 2

- Consider the equation $f = "3 + 2x + 2y + 3xy + y^2 + x^2"$
- we can visualise this in $\mathbb{R}^3$.



b) “$3 + 2x + 2y + 3xy + y^2 + x^2$”



# Generalisation of Functions to N variables

- In order to write these functions in a more invariant way, we fix the following notation:

  $M = \mathbb{Z}^n,\ M_{\mathbb{R}} = \mathbb{Z}^n \otimes \mathbb{R},\ N = Hom_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}),$ and $N_{\mathbb{R}} = N \otimes \mathbb{R}$

- Then, the following function

$$
\begin{aligned}
f :& M_{\mathbb{R}} \to \mathbb{R} \\
& z \mapsto f(z) = " \sum_{n \in S} a_n z^{n}"
\end{aligned}
$$

  can be reexpressed as:

$$
f(z) := min\{a_n + < n, z > | n \in S\},
$$

  where $S$ is a finite subset of $N$.

# Example of Generalisation

- Consider the function $f = "1 + (0 \cdot x) + (0 \cdot x^2) + (2 \cdot x^3)"$
- Then,

$$M = \mathbb{Z},\ M_{\mathbb{R}} = \mathbb{Z} \otimes \mathbb{R} \cong \mathbb{R},\ N = Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}),\ \text{and } N_{\mathbb{R}} = N \otimes \mathbb{R}$$

- Note that $N$ can be viewed as a 1-dimensional vector space spanned by the projection onto $x$-axis function, $Pr_x$
- We denote the evaluation of $n \in N$ on $m \in M$ by <n,m>
- So, the function above can be viewed as:

$$f = min\{1, 0 + x, 0 + 2x, 2 + 3x\}$$

or

$$f = min\{1 + < 0Pr_x, x >, 0 + < Pr_x, x >, 0 + < 2Pr_x, x >, 2 + < 3Pr_x, x >\},$$

where $S$ is the finite subset $\{0, Pr_x, 2Pr_x, 3Pr_x\}$ of $N = Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$.

# Polyhedron

- A polyhedron $\sigma$ in $M_{\mathbb{R}}$ is a finite intersection of closed half-spaces (A hyperplane divides its affine space into 2. Any of these 2 parts in the affine plane is called a half space.). A face of a polyhedron is a subset given by the intersection of $\sigma$ with a hyperplane $H$ such that $\sigma$ is contained in a half-space with boundary $H$.
- The boundary of $\delta(\sigma)$ of $\sigma$ is the union of all proper faces of $\sigma$, and the interior $Int(\sigma)$ is $\sigma \backslash \delta(\sigma)$.
- The polyhedron $\sigma$ is a lattice polyhedron if its an intersection of half-spaces defined over $\mathbb{Q}$ and all the vertices lie in $M$.
- A polytope is a compact polyhedron.

# Newton Polytope

- We now explain a simple way way to see what $V(f)$, the tropical hyper surface, looks like
- Given $f = \text{"}\sum_{n \in S} a_n z^n\text{"}$, we define the newton polytope of $S$ to be:

$$\Delta(S) = Conv(S) \subset N_{\mathbb{R}}$$

that is to say, the convex hull of $S$ in $N_{\mathbb{R}}$

# Newton Polytope Continued

- The coefficients $a_n$ then define a function

$$\psi : \Delta_S \to \mathbb{R}$$

as follows.
- We define the upper convex hull $\tilde{S} = \{(n, a_n) | n \in S\} \subset N_{\mathbb{R}} x \mathbb{R}$
- Namely,

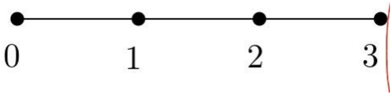
$$\tilde{\Delta}_S = \{(n, a) \in N_{\mathbb{R}} x R | \text{ there exists } (n, a^{'}) \in \text{ Conv}(\tilde{S}) \text{ with } a \geq a^{'}\}$$

- We then define

$$\psi(n) = min\{a \in \mathbb{R} | (n, a) \in \tilde{\Delta}_S\}.$$

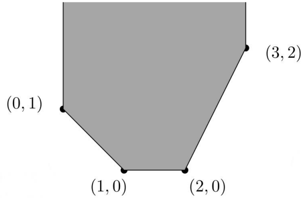
# Example of Convex Hull

- Let's take a look at this function $f = "1 + (0 \cdot x) + (0 \cdot x^2) + (2 \cdot x^3)"$ again
- we have established $S$ is the finite subset $\{0, Pr_x, 2Pr_x, 3Pr_x\}$ of $N = Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$
- So, the set $S$ is a set of discrete points in the 1 dimensional vector space
- Thus, the convex hull of this discrete set is



# Example

- Recall, we define the upper convex hull as $\tilde{S} = \{(n, a_n) | n \in S\} \subset N_{\mathbb{R}} x \mathbb{R}$ and $\tilde{\Delta}_S = \{(n, a) \in N_{\mathbb{R}} x R |$ there exists $(n, a^{'}) \in \text{Conv}(\tilde{S})$ with $a \geq a^{'}\}$
- Since $S = \{0, 1, 2, 3\}$ and the corresponding coefficients of this points are $\{1, 0, 0, 2\}$, our set $\tilde{S} = \{(0, 1), (1, 0), (2, 0), (3, 2)$, which lives in $\mathbb{R}^2$
- Thus, our upper half convex plane, $\tilde{\Delta}_S$, looks like:



# Polyhedral Decomposition

- A (lattice) polyhedral decomposition of a lattice polyhedron $\Delta \subset N_{\mathbb{R}}$ is a set $\mathbb{P}$ of (lattice) polyhedra in $N_{\mathbb{R}}$ called cells such that:
  - $\Delta = \cup_{\sigma \in \mathbb{P}} \sigma$
  - If $\sigma \in \mathbb{P}$ and $\tau \subset \sigma$ is a face, then $\tau \in \mathbb{P}$
  - If $\sigma_1, \sigma_2 \in \mathbb{P}$, then $\sigma_1 \cap \sigma_2$ is a face of both $\sigma_1$ and $\sigma_2$.
- For a polyhedral decomposition $\mathbb{P}$ of $\Delta_S$, denote by $\mathbb{P}_{max}$ the subset of maximal cells of $\mathbb{P}$.
- To get a polyhedral decomposition of $\mathbb{P}$ of $\Delta_S$, we just take $\mathbb{P}$ to be the set of images under the projection $N_{\mathbb{R}} x \mathbb{R} \to N_{\mathbb{R}}$ of proper faces of $\tilde{\Delta}_S$. A $\mathbb{P}$ of $\Delta_S$ obtained in this way from the graph of a convex piecewise linear function is called a regular decomposition, and these decompositions play an important role in the combinatorics of convex polyhedra.

# Definition of Discrete Legendre Transform

- The Discrete Legendre Transform of $(\Delta_S, \mathbb{P}, \psi)$ is the triple $(M_{\mathbb{R}}, \tilde{\mathbb{P}}, \tilde{\psi})$ where:

$$\tilde{\mathbb{P}} = \{\tilde{\tau} : \tau \in \mathbb{P}\}$$

with

$$\tilde{\tau} = \{m \in M_{\mathbb{R}} | a \in \mathbb{R} \text{ such that } < -m, n > +a \leq \psi(n) \forall n \in \Delta_S, \text{ with equality for n} \in \tau\},$$

and
$\tilde{\psi}(m) = max\{a| < -m, n > +a \leq \psi(n) \forall \text{ n} \in \Delta_S$.

Thank You for Your Attention!