Measured Foliations

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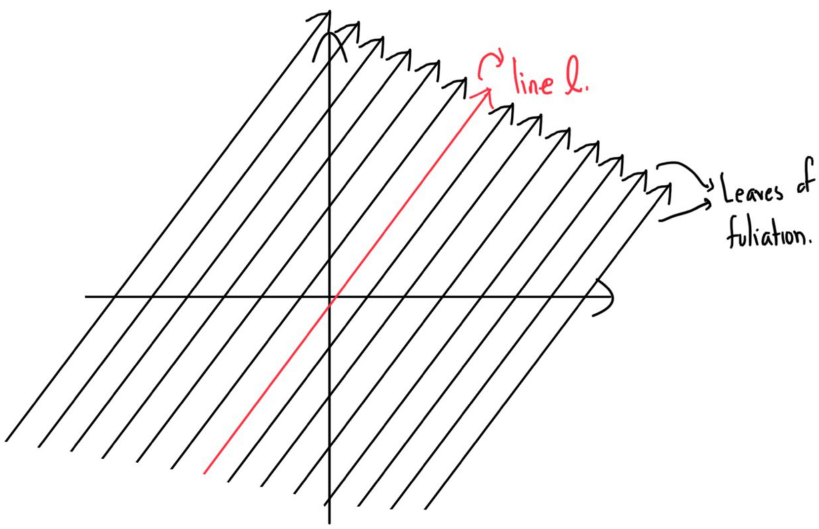
In this set of notes, I will introduce and prove Thurston's Construction. Before doing that, I will briefly recap measured foliations and their construction.

1 Measured Singular Foliation on the Torus

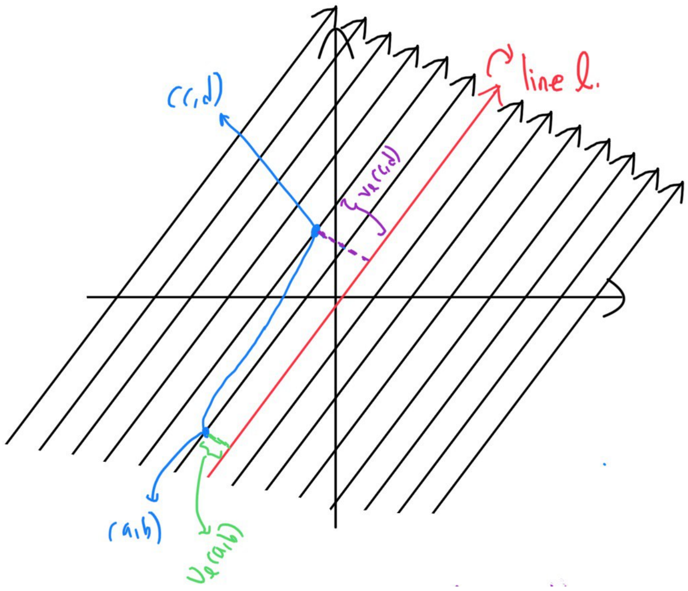
We will first focus our attention on the simple case of the torus before providing the general definition of a measured foliation. I will also elaborate on what it means for a linear map of the torus to stretch the torus along one foliation and shrink along the other.

Let l be any line passing through the origin in $\mathbb{R}^2$. The line l determines a foliation $\widetilde{F}_l$ of $\mathbb{R}^2$, which consists of all lines in $\mathbb{R}^2$ parallel to l. Translations of $\mathbb{R}^2$ takes lines to lines, and so any translation preserves $\widetilde{F}_l$, meaning that leaves are sent to leaves.

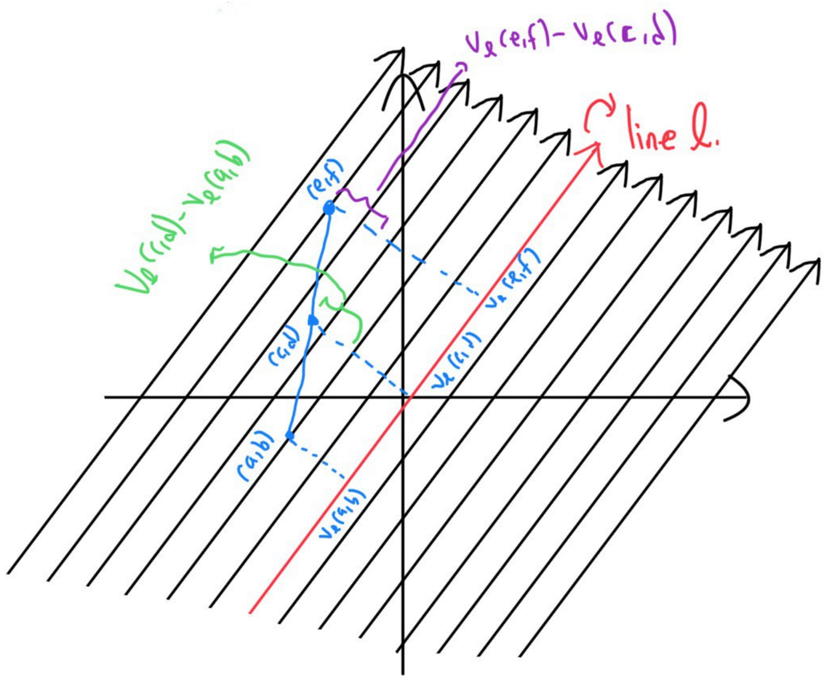


Since all deck transformations for the standard covering $\mathbb{R}^2 \to T^2$ are translations, the foliation $\widetilde{F}_l$ descends to a folation F_l of T^2.

There is an additional structure we will equip the foliations $\widetilde{F}_l$ with. Let $v_l : \mathbb{R}^2 \to \mathbb{R}$ be the function that records distance from any point in $\mathbb{R}^2$ to l. In this picture, the arc from (a, b) to (c, d) denotes a transverse arc in $\mathbb{R}^2$:



v_l will allow us to calculate the difference in height between different points in $\mathbb{R}^2$ with respect to the line l.



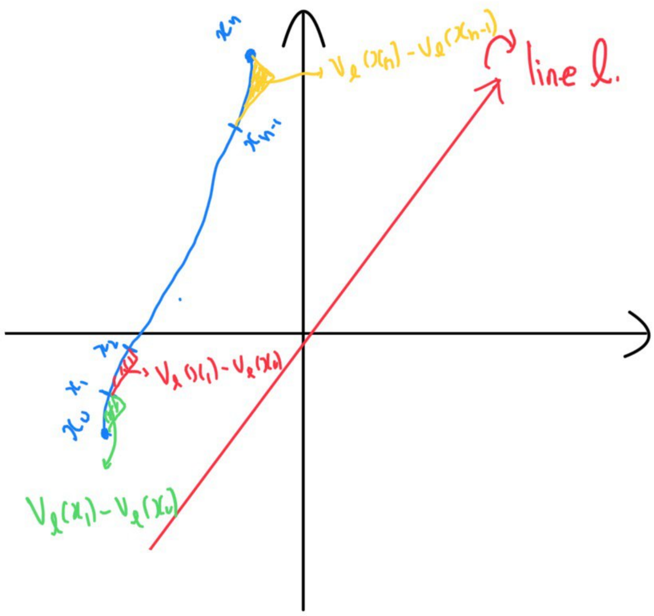
This in turn will let us define a measure in $\mathbb{R}^2$.

The measure μ with respect to the foliation is a function that associates each smooth arcs transverse to foliation F to a real number.

$$\mu : \{\text{Smooth Arcs transverse to foliation } \widetilde{F_l}\} \mapsto \mathbb{R}$$

Integration against the 1-form dv_l provides a transverse measure on $\widetilde{F_l}$. This means that any smooth arc α transverse to the leaves of $\widetilde{F_l}$ can be assigned a length defined by $\mu(\alpha) = \int_\alpha dv_l$. The quantity $\mu(\alpha)$ is the total variation of α in the direction perpendicular to l.

Let's break down what this means. Let $P = \{x_0, x_1, ..., x_n\}$ denote the partition of the smooth arc α transverse to foliation F.



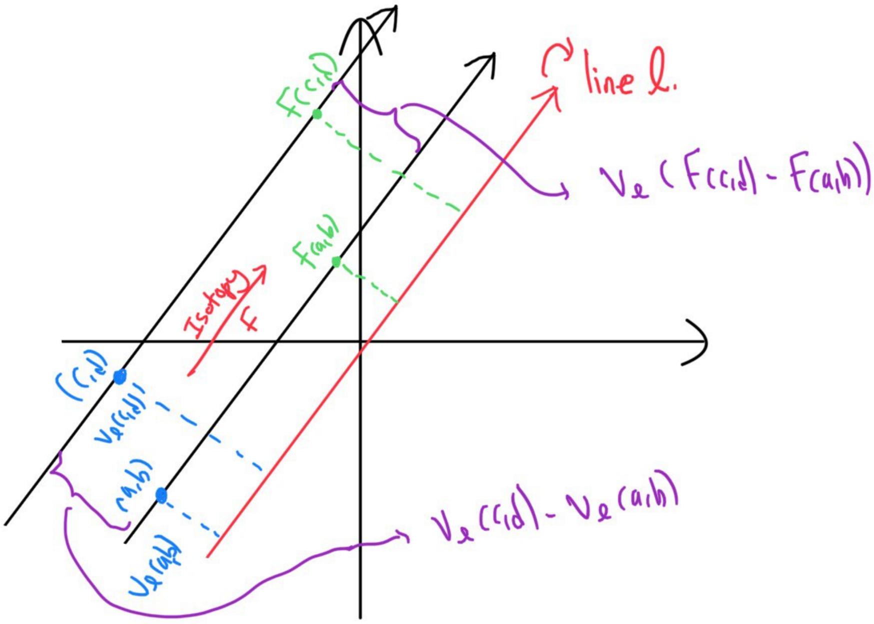
The measure of a smooth arc α transverse to F is defined as shown below:

$$\mu(\alpha) = \sup_P \sum_{i=0}^{n-1} |v_l(x_{i+1}) - v_l(x_i)|,$$

where P is the collection of all possible partitions of α.

So, intuitively, we can see that each smooth arc transverse to the foliation is given a real number which measure the total oscillation from the start point to the end point.

Now, we shall describe some properties of the measure. Firstly, note that $\mu(\alpha)$ is invariant under isotopies of α that move each point α within the leaf of $\widetilde{F}_l$ in which it is contained.



The reason for this is that if a point only shifts within the leave it is contained in, there is no change in height with respect to the base line l (Note that each leaf in the foliation is parallel to l). Since μ can be intepreted as the integration against the 1-form dv_l, this tells us that the measure is indeed invariant under isotopies of α that move each point α within the leaf of $\widetilde{F}_l$ in which it is contained.

Next, The 1-form dv_l is preserved by translations. This is due to similar reasoning above. As the entire plane is shifted by the translation, at if a point only shifts within the leave it is contained in, there is no change in height with respect to the base line l. So, since μ can be interpreted as the integration against the 1-form dv_l, the 1-form dv_l is preserved. So, the 1-form dv_l descends to a 1-form w_l on T^2 and induces a transverse measure on the foliation F_l. The structure of a foliation on T^2 together with a transverse measure is called a transverse measured foliation on T^2.

1.1 Worked out examples of foliation of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

The characteristic polynomial of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is:

$$\lambda^2 - tr(A)\lambda + det(A) = 0.$$

Since $tr(A) = 2 + 1 = 3$ and $A \in SL(2, \mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}$, we have:

$$\lambda^2 - 3\lambda + 1 = 0.$$

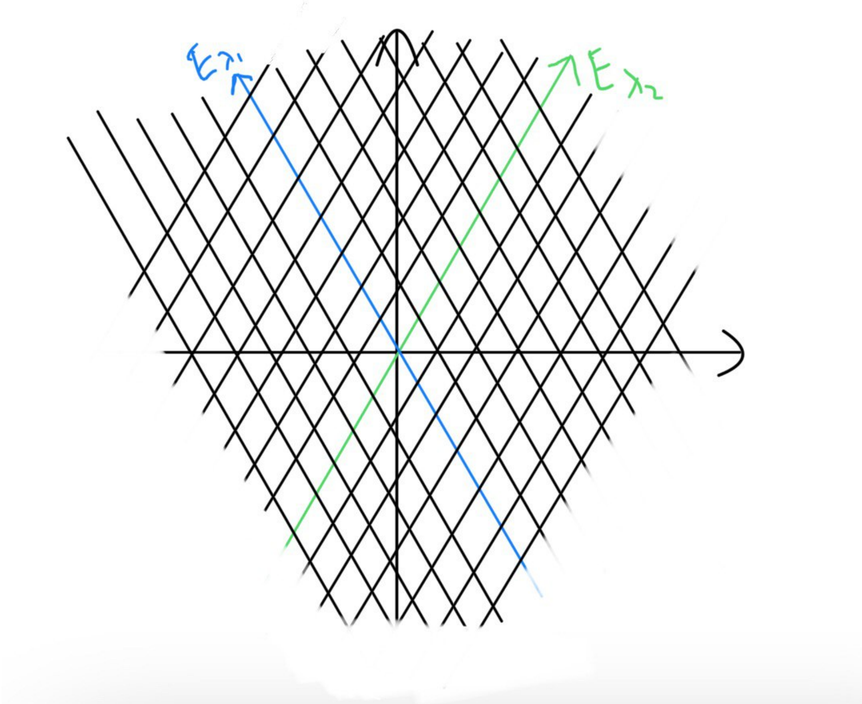
Thus, we have the following eigenvalues:

$$\lambda_1 = \frac{-\sqrt{5} + 1}{2}$$
$$\lambda_2 = \frac{\sqrt{5} + 1}{2}.$$

The corresponding eigenspace of λ_1 and λ_2, denoted as E_{λ_1} and E_{λ_2} respectively, are:

$$E_{\lambda_1} = \{m \begin{bmatrix} \frac{-\sqrt{5}+1}{2} \\ 1 \end{bmatrix} : m \in \mathbb{R}^2\}$$
$$E_{\lambda_2} = \{n \begin{bmatrix} \frac{\sqrt{5}+1}{2} \\ 1 \end{bmatrix} : n \in \mathbb{R}^2\}$$

This results in the foliations on T^2:



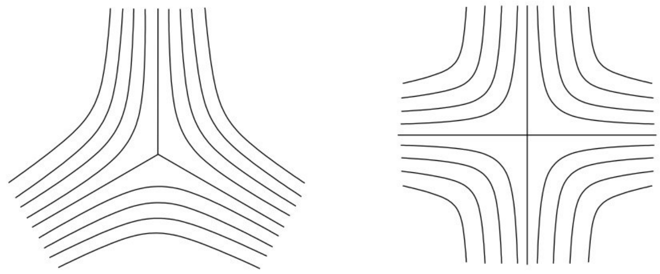
1.2 Measured Singular Foliations on $\Sigma_{g,n}$

On a higher genus surface, it is not clear what it means for a homeomorphism to stretch in the direction of a single vector. To counter this, we can construct measured foliation on a higher genus surface, which we will then map it to $\mathbb{R}^2$ using smooth charts. This will allow us to embed the foliations in $\mathbb{R}^2$, where we can see how a homeomorphism to stretch the surface in the direction of that foliation.

A singular foliation F on a closed surface $\Sigma_{g,n}$ is a decomposition of $\Sigma_{g,n}$ into a disjoint union of subsets of $\Sigma_{g,n}$, called the leaves of F, and a finite set of points of $\Sigma_{g,n}$, called singular points of F, such that the following 2 conditions hold:

A. For each non-singular point $p \in \Sigma_{g,n}$, there is a smooth chart from a neighborhood of p to R^2 that takes leaves to horizontal line segments. The transition maps between any 2 of these charts are smooth maps of the form $(x, y) \mapsto (f(x, y), g(y))$. In other words, the transition maps take horizontal lines to horizontal lines.

B. For singular points $p \in \Sigma_{g,n}$, there is a smooth chart from a neighborhood of p to $\mathbb{R}^2$ that takes leaves to level sets of a k-pronged saddle, $k \geq 3$.



Just like how we gave foliations on the torus a measure, we want the foliations on higher genus surfaces to be equipped with the transverse measure too. Which is to say a length function defined on arcs transverse to the foliation. However, first, we need to define leaf-preserving isotopies first.

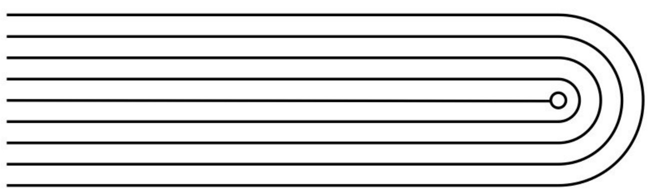
Let F be a foliation on a surface $\Sigma_{g,n}$. A smooth arc α in $\Sigma_{g,n}$ is transverse to F if α misses the singular points of F and is transverse to each leaf of F at each of its interior point. Let $\alpha, \beta : \mathbb{I} \to \Sigma_{g,n}$ be smooth arcs transverse to F. A leaf preserving isotopy from α to β is a map $H : \mathbb{I}$ x $\mathbb{I} \to \Sigma_{g,n}$ such that:

- $H(\mathbb{I}$ x $\{0\} = \alpha)$ and $H(\mathbb{I}$ x $\{1\} = \beta)$
- $H(\mathbb{I}$ x $\{t\}$ is transverse to F for each $t \in [0, 1]$.
- $H(\{0\}$ x $\mathbb{I}$ and $H(\{1\}$ x $\mathbb{I}$ are each contained in a single leaf.

A transverse measure μ on a foliation F is a map that assigns a positive real number to each smooth arc transverse to F, so that μ is invariant under leaf-preserving isotopy and μ is regular with respect to Lebesgue measure. This means that each point of $\Sigma_{g,n}$ has a neighborhood U and a smooth chart $U \to \mathbb{R}^2$ so that the measure μ is induced by $|dy|$ on $\mathbb{R}^2$.

Thus, a measured foliation (F, μ) on a surface S is a foliation F of S equipped with a transverse measure μ.

Punctures and Boundary At a puncture, a foliation takes the form of a regular point or a k-pronged singularity with $k \geq 3$, as in the case of foliations on closed surfaces. At a puncture, however, we can allow a one prong singularity.



A measured foliation on a compact surface S with nonempty boundary is defined similarly to the case when S is closed. There are four different pictures in the neighborhood of a point in the boundary of S depending on whether or not the point is singular and whether or not the leaves are parallel to the boundary or transverse to the boundary.