Introduction to Projective Representations

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Abstract

In this paper, I will first introduce key definitions that will be used later on. Next, I will introduce 2 definitions of projective representations and prove their equivalence and then, draw the link to "ordinary" representations. Lastly, we will look at an example of projective representations of K_4.

1 Introduction

Definition 1.1 (Transversals) *Let G and H be arbitrary groups. If $K \leq G$, then a (right) transversal of K in G (or a complete set of right coset representatives) is a subset T of G consisting of one element from each right coset of K in G. Also, if $\pi : G \to H$ is surjective, then a lifting of $x \in H$ is an element $l(x) \in G$ with $\pi(l(x)) = x$. In this case, the function $l; H \to G$ is also called a right transversal.*

Note that if T is the right transversal, then G is the disjoint union $G = \cup_{t \in T} K + t$. Thus, every element $g \in G$ has an unique factorisation $g = k + t$ for $k \in K$ and $t \in T$.

Definition 1.2 (The Factor Set) *If $\pi : G \to Q$ is a surjective homomorphism with kernel K, and if $l : Q \to G$ is the transversal with $l(1) = 0$, then the function $f : Q$ x $Q \to K$, defined as $(x, y) \mapsto l(x) + l(y) - l(xy)$, is called a factor set. (The factor set is dependent on the transversal l.)*

Definition 1.3 (Inertia group of θ in G) *Let $H \trianglelefteq G$ and let $\theta \in Irr(H)$, where $Irr(H)$ is the set of all irreducible $\mathbb{C}$-character of H. Then,*

$$I_G(\theta) = \{g \in G : \theta^g = \theta\}$$

is the inertia group of θ in G. θ^g is the character from H to $\mathbb{C}$ defined as:

$$\theta^g : h \mapsto \theta(ghg^{-1}).$$

Definition 1.4 (Invariant character of G) *If $\theta \in Irr(G)$ and $G = I_G(\theta)$, then θ is said to be invariant in G.*

Definition 1.5 (Projective Representation) *A Projective Representation of a group Q is a homomorphism*

$$\tau : Q \to PGL(n, \mathbb{C}) = \frac{GL(n, \mathbb{C})}{Z(n, \mathbb{C})}.$$

where $Z(n, \mathbb{C})$ denotes the center of the group $GL(n, \mathbb{C})$, $Z(GL(n, \mathbb{C}))$.

Note that $Z(n, \mathbb{C}) = \{\lambda \mathbb{I}_n : \lambda \in \mathbb{C}^*\}$, where $\mathbb{I}_n$ is the identity matrix. This is because if a matrix A is in the center, then it must commute with every invertible matrix. In particular it must commute with elementary matrices which corresponds to the operations on row or columns, depending on the multiplication is made on the left or on the right. In particular if the elementary matrix is B, denoting the multiplication by $a \in \mathbb{F}$ of the i^{th} row, then you have $AB = BA$; hence multiplying the i^{th} row by $a \in \mathbb{F}$ is equivalent to multiplying the i^{th} column by the same scalar. Hence, a matrix in the center must be diagonal. Viceversa, as you already noticed, a diagonal matrix is in the center.

Furthermore, $\{\lambda \mathbb{I} : \lambda \in \mathbb{C}^*\} \cong \mathbb{C}^*$. Thus, we can rewrite the definition of projective representation as:

$$\tau : Q \to PGL(n, \mathbb{C}) = \frac{GL(n, \mathbb{C})}{\mathbb{C}^*}.$$

There is an equivalent definition of projective representations and we shall prove it as a proposition.

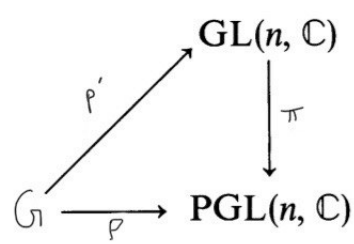
Theorem 1.1 *Let P be the projective representation of G on a complex vector space of dimension n. The, there exist the following maps:*

$$P' : G \to GL_n(\mathbb{C})$$
$$\rho : GxG \to \mathbb{C}^*$$

such that

$$P'(g)P'(h) = \rho(g, h)P'(gh), \forall g, h \in G.$$

Conversely, if there are maps $P' : G \to GL_n(\mathbb{C})$ and $\rho : GxG \to \mathbb{C}^*$ satistfying $P'(g)P'(h) = \rho(g, h)P'(gh), \forall g, h \in G$, then there exist an unique homomorphism $P : G \to PGL_n(\mathbb{C})$ such that $P(g) = \pi P'(g), \forall g \in G$.



proof 1 *Let X be the transversal of $Z(GL_n(\mathbb{C}))$ in $GL_n(\mathbb{C})$ (ie $X \subset GL_n(\mathbb{C})$), and define $P' : G \to GL_n(\mathbb{C})$ by setting $\forall g \in G, P'(g)$ as the unique element of X such that $\pi P'(g) = P(g)$, that is to say $P'(g) = \pi^{-1}P(g)$.*
Now, let $g, h \in G$. Then, we have $P(gh) = P(g)P(h)$ as P is a projective representation. Using the commutative diagram above, we have $P'(gh)\mathbb{C}^* = P'(g)P'(h)\mathbb{C}^*$, which implies there is a unique $\rho(g, h) \in \mathbb{C}^*$ such that $P'(gh) = P'(g)P'(h)$.
Conversely, if we have the maps $P' : G \to GL_n(\mathbb{C})$ and $\rho : GxG \to \mathbb{C}^*$ satistfying $P'(g)P'(h) =$

$\rho(g,h)P'(gh), \forall g,h \in G$, we define the projective representation $P : G \to PGL_n(\mathbb{C})$ as $P = \pi P'$. Then,

$$\begin{aligned} P(gh) &= \pi(P'(gh)) \\ &= \pi(\rho(g,h)^{-1}P'(g)P'(h)) \\ &= \pi(P'(g)P'(h)) \\ &= \pi(P'(g))\pi(P'(h)) \\ &= P(g)P(h), \forall g,h \in G. \end{aligned}$$

This is possible as π is the canonical standard surjective homomorphism from $GL_n(\mathbb{C}) \to PGL_n(\mathbb{C})$. Therefore, P is an group homomorphism from G to $PGL_n(\mathbb{C})$, ie a projective representation.

The theorem gives a new way to see a projective representation which is equivalent. If we have two maps $P' : G \to GL_n(\mathbb{C})$ and $\rho : GxG \to \mathbb{C}^*$ satisfying $P'(g)P'(h) = \rho(g,h)P'(gh), \forall g,h \in G$, we will call P' a projective representation.

The function $\rho : GxG \to \mathbb{C}^*$ is the factor set of $\pi : GL_n(\mathbb{C}) \to PGL_n(\mathbb{C})$, as the definition aboved is satisfied. The $Ker(\pi) = \{\lambda \mathbb{I} : \lambda \in \mathbb{C}^*\} \cong \mathbb{C}^*$. Then, $l : PGL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is defined such as each coset in the quotient group is mapped to it's representative, which belongs to the set of $GL_n(\mathbb{C})$. The identity coset will be the identity matrix $\mathbb{I}$ as it's representative. Thus, $l(\mathbb{I}\mathbb{C}^*) = \mathbb{I}$.

Projective Representations arise in a natural way when we try to answer the following question: Given an irreducible representation ϕ of a normal subgroup $H \trianglelefteq Q$, can the representation of H be extended to a representation of Q? We will show how right now.

Theorem 1.2 *Let $N \trianglelefteq G$ and suppose η is an irreducible $\mathbb{C}$-representation of N whose character is invariant in G. Then, there exist a projective $\mathbb{C}$-representation χ of G such that $\forall n \in N$ and $g \in G$, we have*

- $\chi(n) = \eta(n)$
- $\chi(ng) = \chi(n)\chi(g)$
- $\chi(gn) = \chi(g)\chi(n)$.

Furthermore, if χ_0 is another projective representation satisfying the 3 conditions above, then $\chi_0(g) = \chi(g)\mu(g)$ for some function $\mu : G \to \mathbb{C}^*$, which is constant on cosets of N.

proof 2 *For $g \in G$ and $n \in N$, we define a new representation as follows:*

$$\eta^g(n) = \eta(gng^{-1}).$$

This map is well-defined as N is a normal subgroup of G. Furthermore, as η affords a G-invariant character, we conclude that η and η^g are similar representations of N.

Now, choose a transversal T for N in G. Take $1 \in T$. For each $t \in T$, choose a nonsingular matrix P_t such that $P_t \eta P_t^{-1} = \eta^t$. Take $P_1 = \mathbb{I}_n$. Since every element of G is uniquely of the form nt and $t \in T$, we can define χ on G by $\chi(g) = \chi(nt) = \eta(n)P_t$.

The first 2 properties are immediate and the last one follows since

$$\begin{aligned}\chi(g)\chi(m) &= \chi(nt)\chi(m) \\ &= \eta(n)P_t\eta(m) \\ &= \eta(ntmt^{-1})P_t \\ &= \chi(ntmt^{-1}t) \\ &= \chi(ntm), \forall g \in G, m \in N\end{aligned}$$

Note that the 3 properties yield:

$$\begin{aligned}\chi(g)\eta(n) &= \chi(gn) \\ &= \chi(gng^{-1}g) \\ &= \eta(gng^{-1})\chi(g)\end{aligned}$$

Thus, we have $\chi(g)\eta(n)\chi(g)^{-1} = \eta(gng^{-1}), \forall g \in G$ *and* $n \in N$.

If A *is any non-singular matrix such that* $A\eta(n)A^{-1} = \eta(gng^{-1}), \forall n \in N$, *then* $A^{-1}\chi(g)$ *commutes with all* $\eta(n)$ *for* $n \in N$ *and thus* $A^{-1}\chi(g)$ *is a scalar matrix by Schur's Lemma.*

If χ_0 *also satisfies the 3 conditions above, we may take* $A = \chi_0(g)$ *and conclude that* $\chi_0(g) = \chi(g)\mu(g)$ *for some* $\mu(g) \in \mathbb{C}^*$.

Also, $\chi(g)\chi(h)\eta(n)\chi(h)^{-1}\chi(g)^{-1} = \chi(g)\eta(hnh^{-1})\chi(g)^{-1} = \eta(ghnh^{-1}g^{-1})$. *Comparing this with* $\chi(gh)\eta(n)\chi(gh)^{-1} = \eta(ghng^{-1}g^{-1})$ *yields* $\chi(g)\chi(h) = \chi(gh)\alpha(g,h)$ *for some* $\alpha(g,h) \in \mathbb{C}^*$. *Thus,* χ *is a projective representation.*

All that remains to check now is that μ *is constant on cosets of* N. *We have*

$$\begin{aligned}\chi(n)\chi(g)\mu(g) &= \chi_0(n)\chi_0(g) \\ &= \chi_0(ng) \\ &= \chi(ng)\mu(ng).\end{aligned}$$

Since $\chi(n)\chi(g) = \chi(ng)$ *is non-singular, the result follows.*

Now, lets look an example of projective representation.

Let K_4 denote the klein 4 group. The klein 4 group is defined as $K_4 = \{1, a, b, ab\}$, with the mulitplication table as below:

	e	a	b	ab
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
ab	c	b	a	e

With the function $\phi : K_4 \to GL_2(\mathbb{C})$ defined as:

$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, ab \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, we have the following multiplication table:

	M(e)	M(a)	M(b)	M(ab)
M(e)	M(e)	M(a)	M(b)	M(ab)
M(a)	M(a)	M(e)	M(ab)	M(b)
M(b)	M(b)	M(ab)	M(e)	M(a)
M(ab)	M(ab)	M(b)	M(a)	M(e)

With the function $\lambda : K_4 \to GL_2(\mathbb{C})$ defined as:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, ab \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, we have the following multiplication table:

	M(e)	M(a)	M(b)	M(ab)
M(e)	M(e)	M(a)	M(b)	M(ab)
M(a)	M(a)	−M(e)	M(ab)	−M(b)
M(b)	M(b)	−M(ab)	M(e)	−M(a)
M(ab)	M(ab)	M(b)	M(a)	M(e)

By comparing the table, the factor set, $f(x, y) : V_4 x V_4 \to \mathbb{C}^*$: is defined as -1 if $(x, y) = (a, a), (a, ab), (b, a)$ and (b, ab) and 1 otherwise.