Cobordisms and Frobenius Algebras

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Summary

1 Abstract and Strategy of Proof

In this set of notes, we will prove the following statement:

Theorem 1.1. *There is a canonical equivalence of categories:*

$$2TQFT_{\mathbb{K}} \cong cFA_{\mathbb{K}}.$$

To prove the statement above, we will break down the categories on the left hand side, $2TQFT_{\mathbb{K}}$ and the right hand side, $cFA_{\mathbb{K}}$. We will define the category of $nCob$ and define the n-dimensional TQFT as a symmetric monodial functor from the category of n cobordisms to the category of $\mathbb{K}$-vector spaces, $\text{Vect}_{\mathbb{K}}$. Then, we will go through the concrete example of $2Cob$, and give a presentation of $2Cob$ in terms of generators and relations. Given the category of 2 cobordisms, $2Cob$, and the category of $\mathbb{K}$-vector spaces, $\text{Vect}_{\mathbb{K}}$, we will see that the functors between the categories, that is the 2-dimensional TQFT, also form a category, the arrows between the objects (2-dimensional TQFT), being monodial natural transformations.
On tangent, we will introduce the category of commutative Frobenius Algebras over $\mathbb{K}$, $cFA_{\mathbb{K}}$, with the morphisms being Frobenius Algebra homomorphisms. Then, we build a $1-1$ correspondence between the 2-dimensional TQFTs and commutative Frobenius Algebras, which is possible as we are given a presentation of 2-dimensional TQFTs in terms of generators and relations.
But before we start everything off, we will kick start with the definition of monodial categories, symmetric monodial categories and monodial functor categories as this is a common theme that appears across both sides of the story.

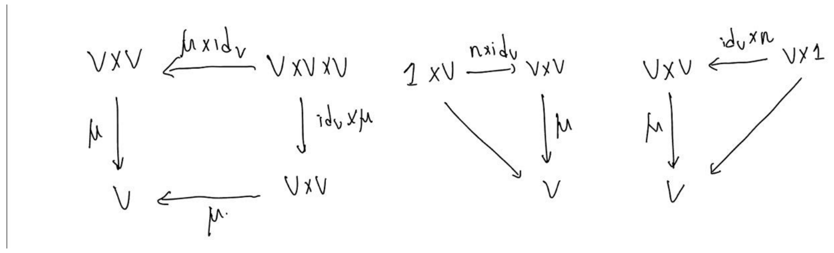
2 Everything Monodial

The important thing to note for this to make sense is that we have the notion of cartesian product of categories. For pair of categories $\mathbf{C}$ and $\mathbf{D}$, there is a category $\mathbf{C}$ x $\mathbf{D}$ defined as follows: its objects are pairs (X, Y) such that X is an object in $\mathbf{C}$ and Y is an object in $\mathbf{D}$. The set of arrows from (X, Y) to (X', Y') is the cartesian product of $\mathbf{C}(X, X')$ x $\mathbf{D}(Y, Y')$. The empty product category (product of zero factors) is denoted as $\mathbf{1}$. It is the category with only a single object, and only a single arrow (the identity arrow of the object).

Definition 2.1 (Monodial Category)**.** *A (strict) monodial category is a category* $\boldsymbol{V}$ *together with two functors:*

$$\mu : \boldsymbol{V} x \boldsymbol{V} \to \boldsymbol{V}, \eta : \boldsymbol{1} \to \boldsymbol{V}$$

satisfying the associativity axiom and the neutral object axiom. Precisely, we require that these three diagrams commute:



The symbol $id_{\boldsymbol{V}}$ stands for the indentity functor $\boldsymbol{V} \to \boldsymbol{V}$, and the diagonal functors without labels are the projections, which are canonical identifications.

We want to emphasis that μ and η are functors, meaning they operate on both objects and arrows:

$$\mathbf{V}x\mathbf{V} \Rightarrow \mu V$$
$$(X, Y) \mapsto X\Box Y$$
$$(f, g) \mapsto f\Box g.$$

So, to each pair of objects, X, Y, a new object $X\Box Y$ is associated and to each pair of arrows, $f : X \to X^{'}$, $g : Y \to Y^{'}$, a new arrow $f\Box g : X\Box Y \to X^{'}\Box Y^{'}$. The fact that $\Box$ is a functor means that compositions and identity arrows are respected. In detail, given compositions $f : X \to X^{'}$, $f^{'} : X^{'} \to X^{''}$, $g : Y \to Y^{'}$, $g^{'} : Y^{'} \to Y^{''}$, then we have:

$$ff^{'}\Box gg^{'} = (f\Box g)(f^{'}\Box g^{'}).$$

This is the equality of arrows $X\Box Y \to X^{''}\Box Y^{''}$. Concerning the identity arrows: given $id_X : X \to X$, and $id_Y : Y \to Y$, we have:

$$id_X\Box id_y = id_{X\Box Y}.$$

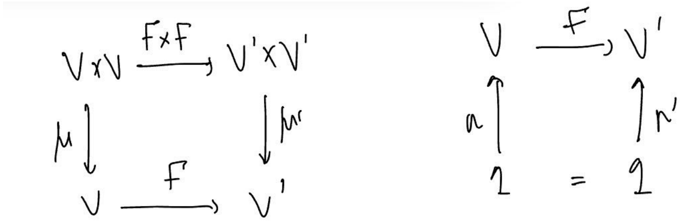
Let I denote the object which is the image of $\eta : \mathbf{1} \to \mathbf{V}$. Then, the statement of the 2 triangular diagrams can be formulated as follows:

$$I\Box X = X = X\Box I, id_I\Box f = f = f\Box id_I,$$

for every object X, and for every arrow f.

We refer to a monodial category by specifying the triple $(\mathbf{V}, \Box, I)$.

Definition 2.2 (Monodial Functor). *A (strict) monodial functor between two (strict) monodial categories* $(\boldsymbol{V}, \Box, I)$ *and* $(\boldsymbol{V}^{'}, \Box^{'}, I^{'})$ *is a functor* $F : \boldsymbol{V} \to \boldsymbol{V}^{'}$ *that commutes with all the structure. Precisely, these* 2 *diagrams are required to commute:*



So, in terms of objects, we have :

$$(XF)\Box^{'}(YF) = (X\Box Y)F, \text{ and } IF = I^{'},$$

and in terms of arrows, we have

$$(fF)\Box^{'}(gF) = (f\Box g)F.$$

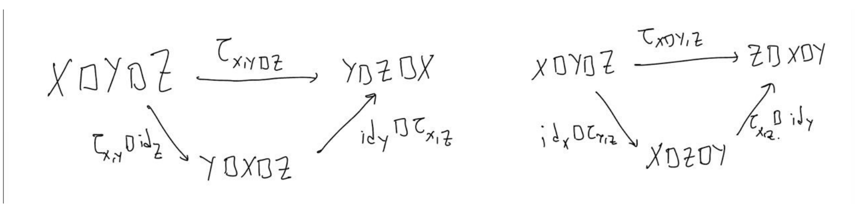
The composition of two monodial functors is again monodial, and that identity functors are monodial. So, all together, there is a category denoted **MonCat** whose objects are the monodial category and the arrows are the functors. Some examples of monodial category that we will see are $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})$, and $(2Cob, \sqcup, \emptyset)$.

Definition 2.3 (Symmetric Monodial Category). *A (strict) monodial category* $(\boldsymbol{V}, \Box, I)$, *is called a symmetric monodial category if for each pair of objects* X, Y, *there is given a twist map:*

$$\tau_{X,Y} : X\Box Y \to Y\Box X,$$

subject to the following 3 *axioms:*

- *The maps are natural*
- *For every triple of objects* X, Y, Z, *these* 2 *diagrams commute:*



- *we have* $\tau_{X,Y}\tau_{Y,X} = id_{X\Box Y}$.

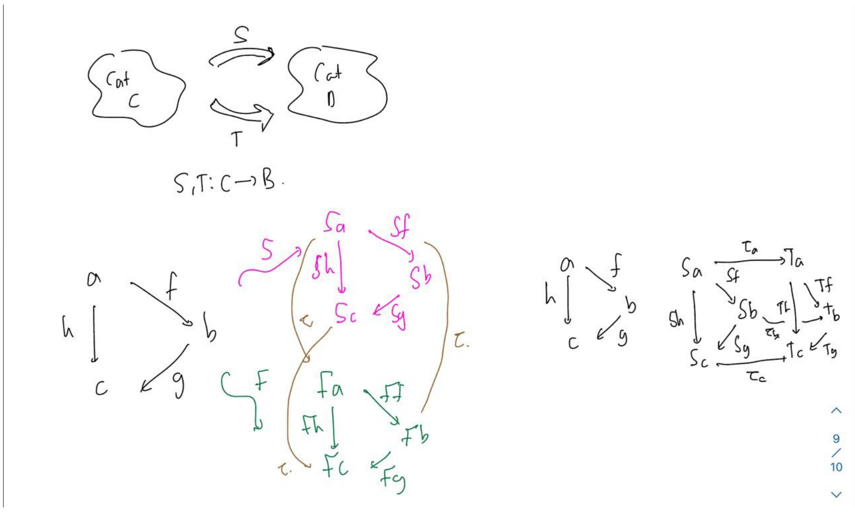
Definition 2.4 (Monodial Natural Transformations)**.** *Let* $(\boldsymbol{V}, \Box, I)$ *and* $(\boldsymbol{V}', \Box', I')$ *be two monodial categories, and let*

$$G, F : \boldsymbol{V} \to \boldsymbol{V}'$$

be two monodial functors. A natural transformation $u : F \to G$ *is called a monodial natural transformation if for every* 2 *objects* X, Y, *in* $\boldsymbol{V}$, *we have*

$$u_X \Box u_Y = u_{X\Box Y},$$

and also $u_I = id_{I'}$.



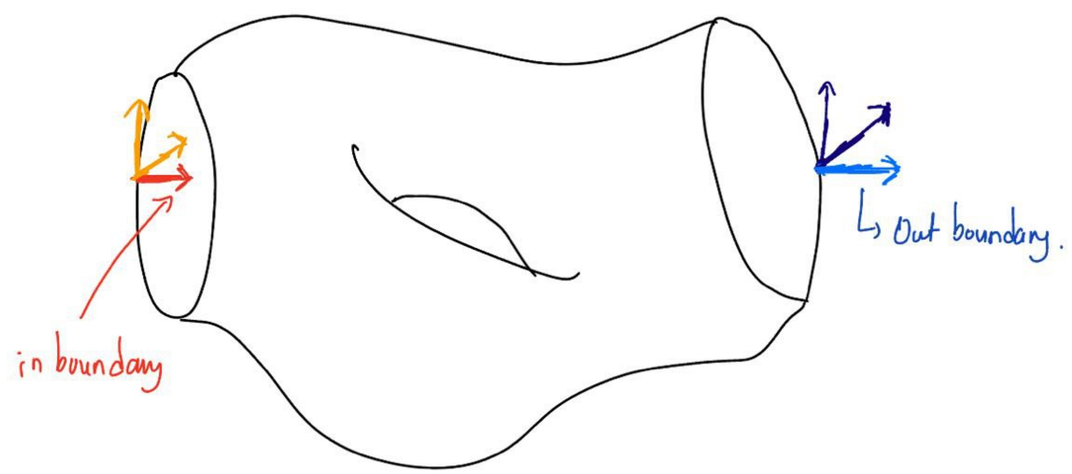
Definition 2.5 (Monodial Functor Categories)**.** *For two fixed monodial categories* $(\boldsymbol{V}, \Box, I)$ *and* $(\boldsymbol{V}', \Box', I')$, *there is a category* $\boldsymbol{MonCat}(\boldsymbol{V}, \boldsymbol{V}')$ *whose objects are the monodial functors from* $\boldsymbol{V}$ *to* $\boldsymbol{V}'$, *and whose arrows are the monodial natural transformations between such functors.*

Definition 2.6 (Symmetric Monodial Functor Categories)**.** *Similarly, given two symmetric monodial categories* $(\boldsymbol{V}, \Box, I, \tau)$ *and* $(\boldsymbol{V}', \Box', I', \tau')$, *there is a category* $\boldsymbol{SymMonCat}(\boldsymbol{V}, \boldsymbol{V}')$ *whose objects are the symmetric monodial functors from* $\boldsymbol{V}$ *to* $\boldsymbol{V}'$, *and whose arrows are the monodial natural transformations between such functors.*

3 Cobordisms

Throughout this set of notes, by the word manifold, I always mean smooth manifold (is differentiable of class C^∞, and unless specified otherwise, we always assume our manifolds to be compact and equipped with an orientation, but we do not always assume our manifolds to be connected. Closed means compact and without boundary. For convenience, we will denote manifolds with boundary by capital roman letters (typically M) while manifolds without boundary are denoted by capital greek letters, like Σ. Manifolds mean smooth manifolds, map between manifolds are always understood to be smooth maps, and maps between manifolds of the same dimension are required to preserve orientation. However, contrary to what is custom in differential topology, we let submanifolds, for example the boundary, to come equipped with an orientation on their own (instead of letting the amibent manifold induce one on it).

Definition 3.1 (In-boundaries and Out-boundaries)**.** *Let Σ be a closed submanifold of M of codimension 1, both equipped with an orientation. At a point $x \in \Sigma$, let $\{v_1, v_2, ..., v_{n-1}\}$ be a positively-oriented basis for $T_x(\Sigma)$. A vector $w \in T_x(M)$ is called a positive normal if $\{v_1, v_2, ..., v_{n-1}, w\}$ is a positively oriented basis for $T_x(M)$. Now, suppose Σ is a connected component of the boundary of M. If a positive normal point inwards, we call Σ an in-boundary and if it points outwards we call it an out-boundary.*



Note that this notion does not depend on the choice of positive normal. Thus, the boundary of a manifold is the union of various in-boundaries and out-boundaries.

3.1 Intuition on Cobordisms

Intuitively, given 2 closed $n-1$-manifolds Σ_0 and Σ_1, a cobordism from Σ_0 to Σ_1 is an oriented n-manifold whose in-boundary is Σ_0 and out-boundary is Σ_1. However, in order to allow cobordisms from a given Σ to itself, we need a more relative description.

An (oriented) cobordism from Σ_0 to Σ_1, where Σ_0 and Σ_1 are closed $n-1$-manifolds, is an (oriented) n-manifold together with the maps:

$$\Sigma_0 \overset{i}{\hookrightarrow} M \overset{i}{\hookleftarrow} \Sigma_1$$

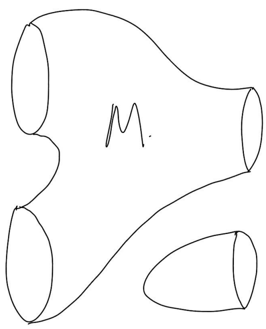
such that Σ_0 maps diffeomorphically onto the in-boundary of M, and that Σ_1 maps diffeomorphically onto the out-boundary of M. So, there exist the following inclusion maps:

$$i_0 : \Sigma_0 \overset{i}{\hookrightarrow} M,$$
$$i_1 : \Sigma_1 \overset{i}{\hookrightarrow} M$$

such that Σ_0 maps diffeomorphically to the in-boundary of M and Σ_1 maps diffeomorphically to the out-boundary of M. We will write this cobordism as:

$$\Sigma_0 \implies \Sigma_1.$$

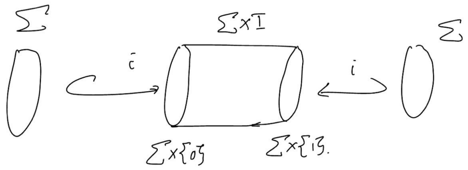
Here is an example of a cobordism from a pair of circles Σ_0 to another pair of circles, Σ_1:



3.2 Cobordisms from Σ to itself

We can get a cobordism from a closed $n-1$ manifold Σ to itself by constructing the cylinder of Σ, Σ x I, and taking the obvious maps:

$$\Sigma \xrightarrow{\sim} \Sigma \text{ x } \{0\} \subset \Sigma \text{ x } I,$$
$$\Sigma \xrightarrow{\sim} \Sigma \text{ x } \{1\} \subset \Sigma \text{ x } I.$$



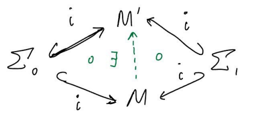
The same construction serves to give a cobordism between any pair of $(n-1)$-manifolds Σ_0 to Σ_1, which are diffeomorphic to Σ:

$$\Sigma_0 \to \Sigma \to \Sigma \text{ x } \{0\} \subset \Sigma \text{ x } I,$$
$$\Sigma_1 \to \Sigma \to \Sigma \text{ x } \{1\} \subset \Sigma \text{ x } I.$$

From the construction above, we can see that any diffeomorphism Σ x $I \to M$ will also define a cobordism $M : \Sigma_0 \implies \Sigma_1$. So, in conclusion, for any 2 diffeomorphic manifolds Σ_0 and Σ_1, there exists a cobordism from Σ_0 to Σ_1. In fact, the cobordism is not unique. One of the main goals of many fields in mathematics is to obtain a classification of the objects of study in their respective field. So, naturally now we will introduce the notion of classification of cobordisms using equivalence relations.

3.3 Equivalent Cobordisms

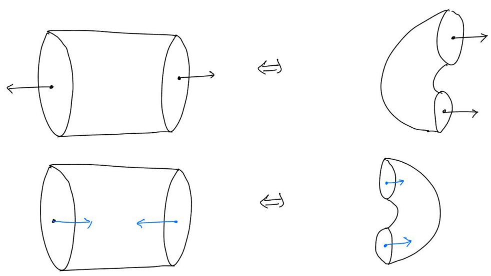
Two cobordisms from Σ_0 to Σ_1, M, M' are called equivalent if there is a diffeomorphism from M to M' making this diagram commute:



Note that the source and target manifolds Σ_0 and Σ_1 are completely fixed, not just up to diffeomorphism.

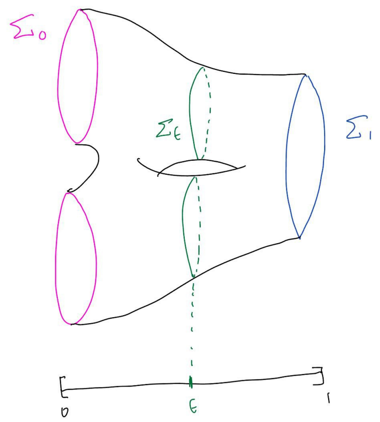
3.4 U-tubes

U-tubes are cylinders with reversed orientation on one of the boundaries: take a closed manifold Σ and map it onto the ends of the cylinder Σ x I, in such a way that both boundaries are in-boundaries (then the out-boundary is empty). We will often draw such a cylinder like this:

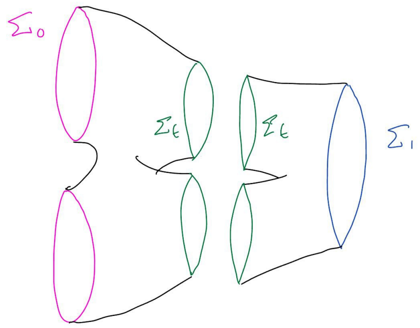


3.5 Decomposition of Cobordisms

An Important feature of a cobordism M is that you can decompose it: this means introducing a sub-manifold Σ which splits M into two parts, with all the in-boundaries in one part and all the out-boundaries in the other; Σ must be oriented such that its positive normal points toward the out-part.



Construction of such a manifold: take a smooth map $f : M \to [0,1]$ such that $f^{-1}(0) = \Sigma_0$ and $f^{-1}(1) = \Sigma_1$, and let Σ_t be the inverse image of a regular value t, oriented such that the positive normal points towards the out-boundaries, just as the positive normal of $t \in [0,1]$ points towards 1. We know that such a map, called the morse map, exists for every smooth compact manifold with boundaries (From John Milnor's Lectures on h-Cobordisms).



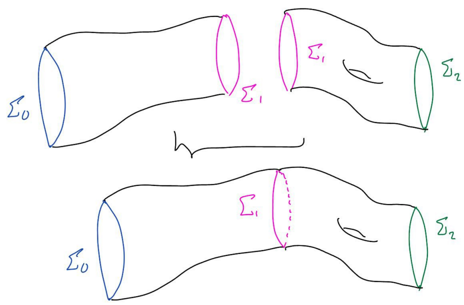
The result is two new cobordisms: one from Σ_0 to Σ_t given by the piece $M_{[0,t]} := f^{-1}([0,t])$ and another from Σ_t to Σ_1 given by the piece $M_{[t,1]} := f^{-1}([t,1])$.

3.6 Giving cobordisms a category

Now, we shall assemble cobordisms as a category in the most natural way possible:

- Objects: closed oriented $(n-1)$-manifolds
- Morphisms: oriented cobordisms as defined earlier

So, we need to show how to compose two cobordisms (and check associativity), and we need to find identity arrows for each object. Given one cobordism $M_0 : \Sigma_0 \implies \Sigma_1$ and another $M_1 : \Sigma_1 \implies \Sigma_2$, then the composition $M_0M_1 : \Sigma_0 \implies \Sigma_2$ should be obtained by gluing together the manifolds M_0 and M_1 along Σ_1. This is a manifold with in-boundary Σ_0 and out-boundary Σ_2, and Σ_1 sits inside it as a submanifold:



However this construction $M_0M_1 := M_0 \sqcup M_1$ is not well defined in the category of smooth manifolds. It is well-defined as a topological manifold, but there is no canonical choice of smooth structure near the gluing locus Σ_1: the smooth structure turns out to be well-defined only up to diffeomorphism. (And not even unique diffeomorphism.)Concerning identity arrows, the identity ought to be a cylinder of height zero, but such a 'cylinder' is not an n-manifold.

3.7 Gluing Cobordism Classes

Both problems are solved by using the following theorem, which tells us that the smooth structure turns out to be well-defined only up to diffeomorphism and hence passing to diffeomorphism classes of cobordisms relative the boundary (fixing the boundary point-wise), we can have a well-defined composition operation on the category of cobordisms (classes).

Theorem 3.1. *Let W be a smooth compact n dimensional manifold with 2 boundary components V_0, V_1 and $W^{'}$ be another smooth compact n dimensional manifold with 2 boundary components $V_1^{'}, V_2^{'}$ and $h : V_1 \to V_1^{'}$ a diffeomorphism. Then, there exists a smooth structure S for $W \cup_h W^{'}$ compatible with the given structure on W and $W^{'}$. S is unique up to a diffeomorphism leaving $V_0, h(V_1) = V_1^{'}$, and $V_2^{'}$ fixed.*

Proof. To prove the existence of the smooth structure, we will use the following lemma:

Theorem 3.2 (Collar Neighborhood Theorem)**.** *Let W be a compact smooth manifold with boundary. There exists a neighborhood of $Bd(W)$ (called a collar neighborhood) diffeomorphic to $Bd(W)$ x $[0, 1)$.*

By the Collar Neighborhood Theorem, there exist collar neighborhoods $U_1, U_1^{'}$ of $V_1, V_1^{'}$ in $W, W^{'}$ and diffeomorphisms $g_1 : V_1$ x $[0, 1) \to U_1$, $g_2 : V_1^{'}$ x $[1, 2) \to U_1^{'}$, such that $g_1(x, 1) = x$, where $x \in V_1$, and $g_2(y, 1) = y$, where $y \in V_1^{'}$. Let $j : W \to W \cup_h W^{'}$, $j^{'} : W^{'} \to W \cup_h W^{'}$ be the inclusion maps in the definition of $W \cup_h W^{'}$. Define a map $g : V_1$ x $(0, 2) \to W \cup_h W^{'}$ by:

$$g(x, t) = j(g_1(x, t)), 0 < t \leq 1$$
$$g(x, t) = j^{'}(g_2(h(x), t)), 1 \leq t < 2.$$

To define a smooth structure ob a manifold, it suffices to define compatible smooth structures on open sets covering the manifold. So, note that $W \cup_h W^{'}$ is covered by $j(W - V_1), j^{'}(W^{'} - V_1^{'})$, and $g(V_1 x(0, 2))$, and the smooth structures defined on these sets by $j, j^{'}$, and g respectively, are compatible. This completes the proof of existence.

Now, we shall show that the smooth structure is unique up to a diffeomorphism leaving $V_0, h(V_1) = V_1^{'}$, and $V_2^{'}$ fixed. We do this by showing that any smooth structure S on $W \cup_h W^{'}$ compatible with the given smooth structure on W and $W^{'}$ is isomorphic to a smoothness structure constructed by pasting together collar neighborhoods of V_1 and $V_1^{'}$ as above. The uniqueness up to diffeomorphism leaving $V_0, h(V_1) = V_1^{'}$, and $V_2^{'}$ fixed, then follows essentially from theorem 6.3 of James Munkres's Elementary Differential Topology. Now, we will use the following theorem:

Theorem 3.3 (The Bicollaring Theorem)**.** *Suppose that every component of a smooth submanifold M of W is compact and two-sided. Then, there exists a bicollar neighborhood of M in W diffeomorphic to M x $(-1, 1)$ in such a way that M corresponds to M x $\{0\}$.*

By the bicollaring theorem, there exists a bicollar neighborhood of U of $j(V_1) = j^{'}(V_1^{'})$ in $W \cup_h W^{'}$ and a diffeomorphism $g : V_1$ x $(-1, 1) \to U$ with respect to the smoothness structure S, so that $g(x, 0) = j(x)$, for $x \in V_1$. Then, $j^{-1}(U \cap j(W))$ and $(j^{'})^{-1}(U \cap j^{'}(W^{'}))$ are collar neighborhoods of V_1 and $V_1^{'}$ in W and $W^{'}$. This completes the proof of uniqueness.

■

So, the theorem above shows that the composition of two cobordisms is a well-defined cobordism class. Clearly, this class only depends on the classes of the two original cobordisms, not on the cobordisms themselves, so we have a well-defined composition for cobordism classes.

Moreover, this composition is associative since gluing of topological spaces (the pushout) is associative. Also, it is easy to prove that the class of a cylinder is the identity cobordism class for the composition law: it amounts to two observations:

- every cobordism has a part near the boundary where it is diffeomorphic to a cylinder.
- the composition of two cylinders is again a cylinder.

3.8 $nCob$, the category

So, given the:

- Objects: closed oriented $(n-1)$-manifolds
- Morphisms: oriented cobordisms classes

and the well-defined operation (gluing) between oriented cobordisms classes, we now have the category of n cobordisms, denoted as $nCob$.

It was mentioned how a diffeomorphism $\phi : \Sigma_0 \to \Sigma_1$ induces a cobordism $C_\phi : \Sigma_0 \implies \Sigma_1$, via the cylinder construction. In fact, this construction is functorial: given two diffeomorphisms $\phi : \Sigma_0 \to \Sigma_1$ and $\psi : \Sigma_1 \to \Sigma_2$, we have

$$C_\phi C_\psi = C_{\phi o \psi}.$$

In particular, a cobordism induced from a diffeomorphism is invertible. Also, the identity diffeomorphism $\Sigma \to \Sigma$ induces the identity cobordism. In other words, the cylinder construction defines a functor from the category of $(n-1)$-manifolds and diffeomorphisms to the category $nCob$.

3.9 Monodial Structure on $nCob$

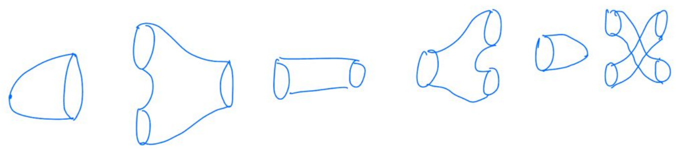
If Σ' and Σ are two $(n-1)$ manifolds then the disjoint union $\Sigma \sqcup \Sigma'$ is again an $(n-1)$ manifold, and given two cobordisms $M : \Sigma_0 \implies \Sigma_1$ and $M' : \Sigma_0' \implies \Sigma_1'$ and, their disjoint union $M \sqcup M'$ is naturally a cobordism from $\Sigma_0 \sqcup \Sigma_0'$ to $\Sigma_1 \sqcup \Sigma_1'$.

The empty n manifold $\emptyset_n$ is a cobordism $\emptyset_{n-1} \implies \emptyset_n$. These structure make $(nCob, \sqcup, \emptyset_n)$ into a monoidal category.

The cobordism induced by the twist diffeomorphism $\Sigma \sqcup \Sigma' \implies \Sigma' \sqcup \Sigma$ will be called the twist cobordism (for Σ and Σ', denoted $T_{\Sigma,\Sigma'}$. It is straight-forward to check that the twist cobordisms satisfy the axioms for a symmetric structure on the monoidal category $(nCob, \sqcup, \emptyset_n)$. This is an easy consequence of the fact that the twist diffeomorphism is a symmetry structure inthe monoidal category of smooth manifolds. So, $(nCob, \sqcup, \emptyset_n)$ is a symmetric monoidal category.

3.10 Generators and relations for $2Cob$

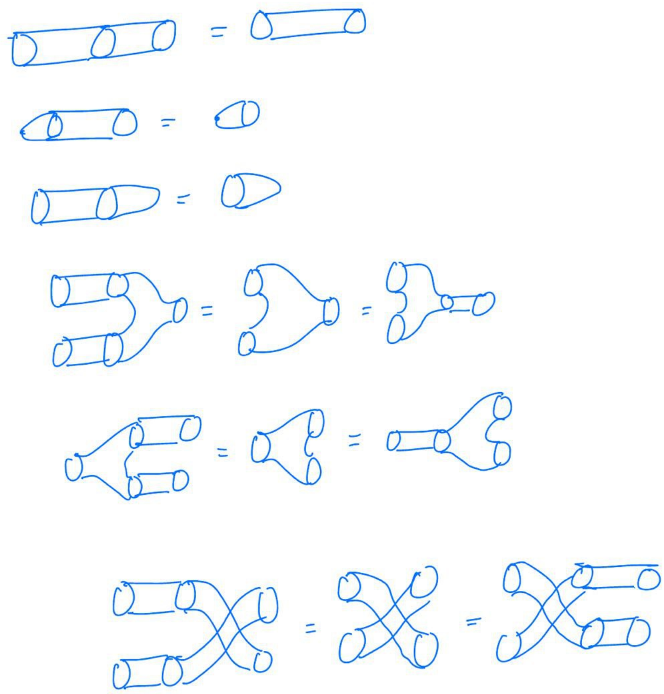
Theorem 3.4. *The monoidal category $2Cob$ is generated under composition (serial connection) and disjoint union (parallel connection) by the following six cobordisms:*



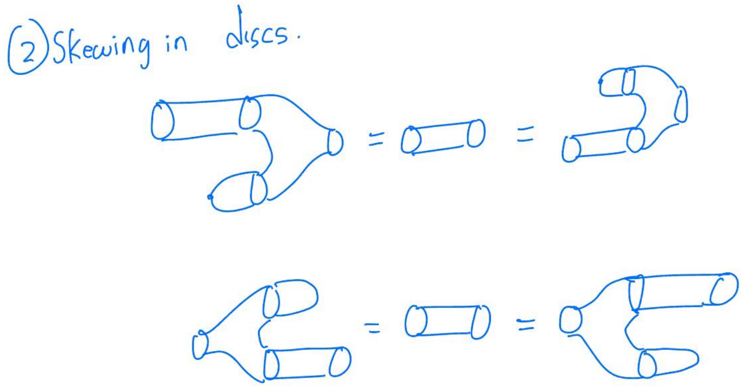
The generators for the $2Cob$ satisfy the following relations:

- Identity Relations:

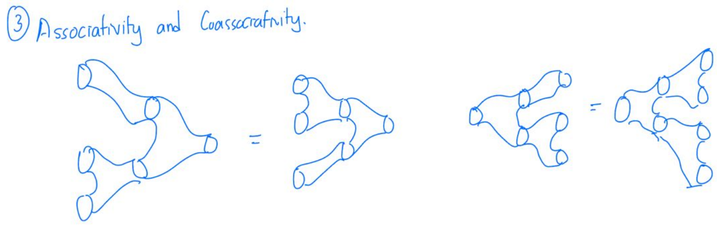
① Cylinders are identites.



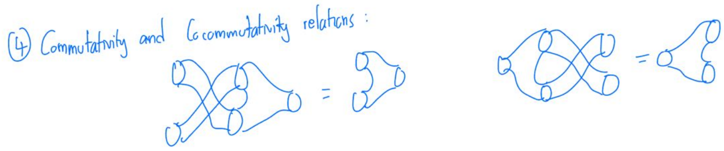
- Skewing in Disc Relations



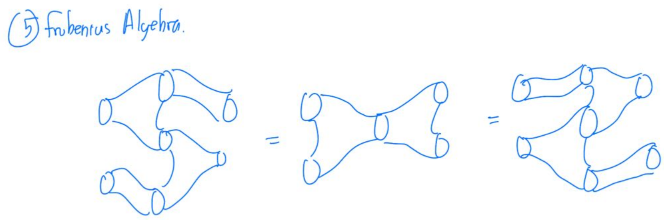
- Associativity and Coassociativity Relations



- Commutativity and Co-communtative Relations



- Frobenius Relations



4 Topological Quantum Field Theories

Roughly, a quantum field theory takes as input spaces and space-times and associates to them state spaces and time evolution operators. The space is modelled as a closed oriented $(n-1)$-manifold, while space-time is an oriented n-manifold whose boundary represents time 0 and time 1. The state space is a vector space (over some ground field $\mathbb{K}$), and the time evolution operator is simply a linear map from the state space of time 0 to the state space of time 1. The theory is called topological if it only depends on the topology of the space-time. This means that 'nothing happens' as long as time evolves cylindrically.

Definition 4.1 (Topological Quantum Field Theory)**.** *An n-dimensional topological quantum field theory (TQFT) is a rule A which to each closed oriented $(n-1)$-manifold Σ associates a vector space $(\Sigma)A$, and to each oriented cobordism $M : \Sigma_0 \to \Sigma_1$ associates a linear map MA from $\Sigma_0 A$ to $\Sigma_1 A$. This rule A must satisfy the following five axioms.*

- *Two equivalent cobordisms must have the same image: $M \cong M' \implies MA = M'A$.*
- *The cylinder $\Sigma x I$, thought of as a cobordism from Σ to itself, must be sent to the identity map of $\Sigma(A)$.*
- *Given a decomposition $M = M'M''$, then $MA = (M'A) \circ (M''A)$ (composition of linear maps).*
- *Disjoint Union goes to tensor product: If $\Sigma = \Sigma' \sqcup \Sigma''$, then $(\Sigma)A = \Sigma'A \otimes \Sigma''A$. This must also hold for cobordisms: if $M : \Sigma_0 \implies \Sigma_1$ is the disjoint union of $M' : \Sigma_0' \implies \Sigma_1'$ and $M'' : \Sigma_0'' \implies \Sigma_1''$, then $MA = M'A \otimes M''A$.*
- *The empty manifold $\Sigma = \emptyset$ must be sent to the ground field $\mathbb{K}$.*

The first two axioms express that the theory is topological: the evolution depends only on diffeomorphism class of space-time. Axiom 4 reflects a standard principle of quantum mechanics: the state space of two independent systems is the tensor product of the two state spaces.

The first three axioms will amount to saying that the rule A is a functor. This is the subject of the next section. Axioms 4 and 5 in turn amount to saying that this functor is furthermore monoidal. Roughly, a monoidal category is one equipped with a 'multiplication' with neutral object. In our case, for manifolds and cobordisms the 'multiplication' is disjoint union, and the neutral object for that operation is the empty manifold. For vector spaces, the 'multiplication' is the tensor product, and the neutral object is the ground field. A monoidal functor is one that preserves such monoidal structure.

Consider the category $\text{Vect}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$ and $\mathbb{K}$-linear maps. Equipped with tensor product as 'paralleling', with the ground field as neutral space, and with the canonical twist map σ which interchanges the two factors of a tensor product, $(\text{Vect}_{\mathbb{K}}, \otimes, \mathbb{K}, \sigma)$ is also a symmetric monoidal category. A monoidal functor (between two monoidal categories) is one that preserves the monoidal structure. A symmetric monoidal functor between two symmetric monoidal categories is one that sends the symmetry of one monoidal category to the symmetry of the other.

Definition 4.2 (Functorial Definition of Topological Quantum Field Theory)**.** *An n-dimensional topological quantum field theory is a symmetric monoidal functor from* $(nCob, \sqcup, \emptyset, T)$ *to* $(Vect_{\mathbb{K}}, \otimes, \mathbb{K}, \sigma)$.

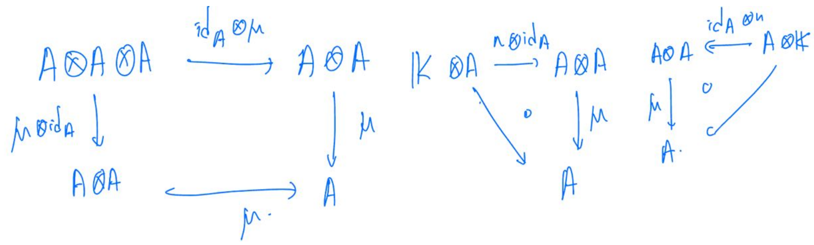
5 Frobenius Algebras

Definition 5.1 (Frobenius Algebra)**.** *A Frobenius algebra is a* $\mathbb{K}$*-algebra A of finite dimension, equipped with a linear functional* $\epsilon : A \to \mathbb{K}$ *whose nullspace contains no nontrivial left ideals. The functional* $\epsilon \in A^*$ *is called a Frobenius form.*

Definition 5.2 ($\mathbb{K}$-Algebra)**.** *A* $\mathbb{K}$*-algebra is a* $\mathbb{K}$*-vector space A together with two* $\mathbb{K}$*-linear maps:*

$$\begin{aligned} \mu : A \otimes A &\to A, \\ \eta : \mathbb{K} &\to A \end{aligned}$$

such that these three diagrams commute:



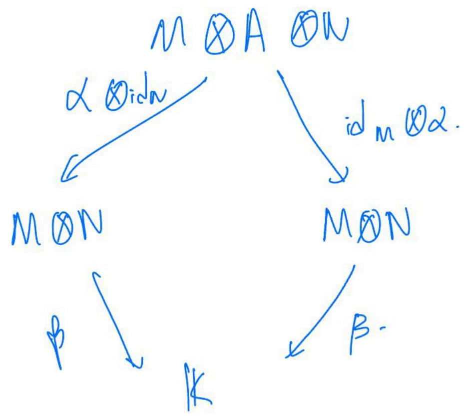
Here, the symbol id_A stands for the identity linear map $A \to A$, and the diagonal maps without labels are scalar multiplication which are canonical isomorphisms.

Every linear functional $\epsilon : A \to \mathbb{K}$ (Frobenius or not) determines canonically a pairing $A \otimes A \to \mathbb{K}$, namely $x \otimes y \to (xy)\epsilon$. Clearly this pairing is associative. Conversely, given an associative pairing $A \otimes A \to \mathbb{K}$, denoted $x \otimes y \to < x|y >$, a linear functional is canonically determined, namely

$$\begin{aligned} A &\to \mathbb{K} \\ a &\mapsto < 1_A | a > = < a | 1_A > . \end{aligned}$$

This shows that there is a one-to-one correspondence between linear functionals on A and associative pairings.

Definition 5.3 (Associate Pairings)**.** *A pairing $M \otimes N \to \mathbb{K}$ is said to be associative if the following diagram commutes:*



In other words, the pairing $x \otimes y \mapsto < x|y >$ is associative when

$$< xa|y > = < x|ay >, \forall x \in M, a \in A, y \in N.$$

Theorem 5.1. *Let $\epsilon : A \to \mathbb{K}$ be a linear functional and let $< x|y >$ denote the corresponding associative pairing $A \otimes A \to \mathbb{K}$. Then the following are equivalent:*

- *The pairing is nondegenerate.*
- *Null(ϵ) contains no nontrivial left ideals.*
- *Null(ϵ) contains no nontrivial right ideals.*

Definition 5.4 (Symmetric Frobenius Algebras)**.** *A Frobenius algebra A is called a symmetric Frobenius algebra if one (and hence all) of the following equivalent conditions holds:*

- *The Frobenius form $\epsilon : A \to \mathbb{K}$ is central; this means that $(ab)\epsilon = (ba)\epsilon$ for all $a, b \in A$.*
- *The pairing $< a|b >$ is symmetric (i.e. $< a|b > = < b|a > \forall a, b \in A$).*

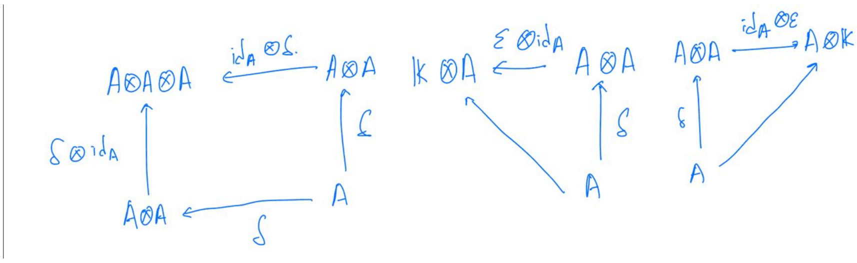
Clearly, commutative frobenius algebras are always symmetric.

5.1 Coalgebras on Frobenius Algebras

The notion of coalgebra over $\mathbb{K}$ is the opposite of the notion of $\mathbb{K}$-algebra, in the sense that the structure maps and the diagrams for their axioms are all just reversed. So a coalgebra over $\mathbb{K}$ is a vector space A together with two $\mathbb{K}$-linear maps:

$$\delta : A \to A \otimes A, \ \epsilon : A \to \mathbb{K},$$

such that the following diagrams commute:



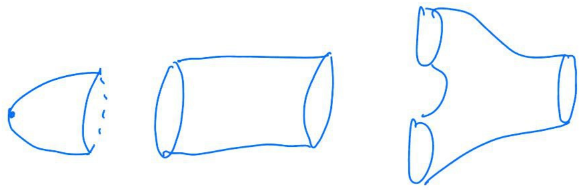
The map δ is called comultiplication, and $\epsilon : A \to \mathbb{K}$ is called the counit (or sometimes the augmentation). The axioms expressed in the diagrams are called coassociativity and the counit condition.

It is not a coincidence that we have denoted the counit by ϵ just like the Frobenius form. The main result of importance to us states that every Frobenius algebra has a unique coalgebra structure for which the Frobenius form is the counit, and which is A-linear and conversely, given a k-algebra, equipped with an A-linear coalgebra structure, then the counit is a Frobenius form. This gives another characterisation of Frobenius algebras – the most important one for our purposes. We will give a quite elementary proof, which does not even involve coordinates. It is based on a graphical calculus.

5.2 Graphical Calculus

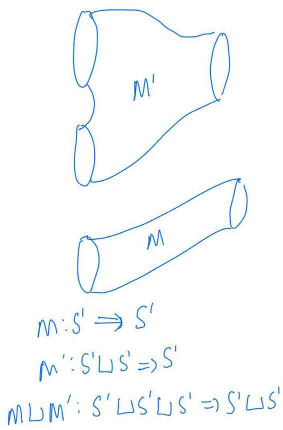
The first observation is that we do not have many pieces to move. If we want to construct a comultiplication on our Frobenius algebra A, all we have to make do with are the following maps: the multiplication $\mu : A \otimes A \to A$, the unit $\eta : \mathbb{K} \to A$, and the Frobenius form $\epsilon : A \to \mathbb{K}$ as well as the Frobenius pairing $\beta : A \otimes A \to \mathbb{K}$, not forgetting the identity map $id_A : A \to A$. These maps come with certain properties which are expressed as commutative diagrams. Our task is to combine these arrows in a natural way to construct a comultiplication, and then combine all the diagrams in order to establish the diagrams that express the properties we want from this comultiplication. . The second observation is that all of these building blocks are maps between tensor powers of A; let A^n denote the tensor product of n copies of A. Of course the ground field appears in the maps, but recall that it is natural to consider $\mathbb{K}$ as the zeroth tensor power of A, the tensor product with zero factors.

Let us first draw the maps that define a $\mathbb{K}$-algebra:

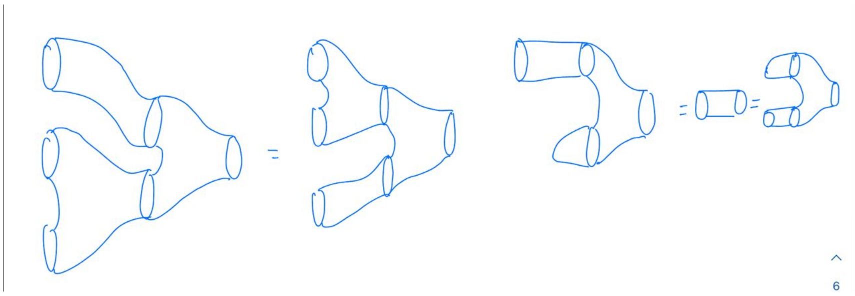


The maps are the unit map, identity map and mulitplication map from left to right.

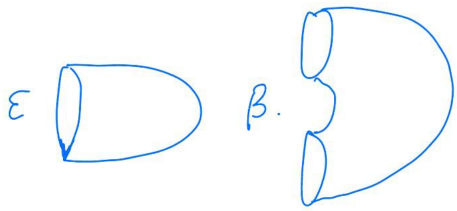
The symbol corresponding to each $\mathbb{K}$-linear map $\phi : A^m \to A^n$ has m boundaries on the left (input holes): one for each factor of A in the source, and ordered such that the first factor in the tensor product corresponds to the bottom input hole and the last factor corresponds to the top input hole. If $m = 0$ we simply draw no in-boundary. Similarly, there are n boundaries on the right (output holes) which correspond to the target A^n, with the same convention for the ordering. The tensor product of two maps is drawn as the (disjoint) union of the two symbols – one placed above the other, in accordance with our convention for ordering. Indeed, the tensor product of two maps is defined by letting the two maps operate independently on their respective arguments, so it is natural that we draw this as two parallel tubes or, in the machine metaphor, as two parallel processes – and similarly for multiple tensor products:



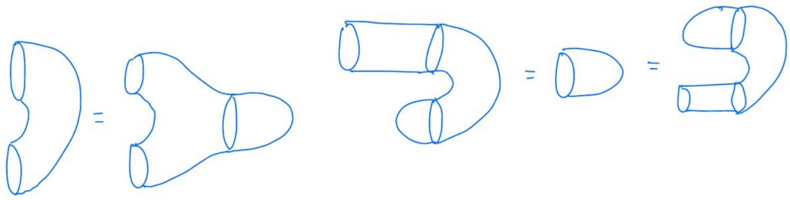
Now, let's express the properties that $\mathbb{K}$-algebra axioms satisfy in terms of graphs.



We now want to express Frobenius structure in graphical language. The linear form $\epsilon : A \to \mathbb{K}$, and the second a bilinear pairing $\beta : A \otimes A \to \mathbb{K}$ are depicted as:



We will draw right away the relation between these 2 maps:



which can be described as $< x|y >= (xy)\epsilon$ and $< 1_A|x >= x\epsilon =< x|1_A >$.

5.3 Category of Frobenius Algebras

A Frobenius algebra homomorphism $\phi : (A, \epsilon) \to (A, \epsilon)$ between two Frobenius algebras is an algebra homomorphism which is at the same time a coalgebra homomorphism. In particular, it preserves the Frobenius form, in the sense that $\epsilon = \phi\epsilon$. Let $FA_{\mathbb{K}}$ denote the category of Frobenius algebras over $\mathbb{K}$ and Frobenius algebra homomorphisms, and let $cFA_{\mathbb{K}}$ denote the full subcategory of all commutative Frobenius algebras.

5.4 Tensor Product of Frobenius Algebras

Given two algebras A and $A^{'}$, consider their tensor product $A \otimes A^{'}$ as vector spaces. Now component-wise multiplication makes $A \otimes A^{'}$ into an algebra:

$$A \otimes A^{'} \otimes A \otimes A^{'} \to A \otimes A^{'}$$
$$(x \otimes y) \otimes (x^{'} \otimes y^{'}) \to xx^{'} \otimes yy^{'}.$$

Note that A only interacts with A, and $A^{'}$ only with $A^{'}$, and that the twist map is crucial in order to construct the new multiplication map from the existing maps.

Also, the tensor product of two coalgebras is again a coalgebra in a natural way. The figures are just the mirror images of those above. The tensor product of two Frobenius algebrasis in a natural way again a Frobenius algebra.

6 Monoids and Monodial Categories

Definition 6.1 (Monoid)**.** *A monoid is a set M with a binary operation (composition law) which is associative and has a neutral element. If we employ infix notation for the composition (with a dot as infix), writing $(a, b) \to a \cdot b$, then the associativity axiom states that for every three elements $a, b, c \in M$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. The neutral element is an element $e \in M$ such that for all $a \in M$ we have $e \cdot a = a = a \cdot e$. It is useful to express this in terms of commutative diagrams. A monoid is a set M together with two functions:*

$$\mu : MxM \to M$$
$$\eta : 1 \to M$$

such that these diagrams commute, similar to the cases above. We will refer to such a monoid by writing the triple (M, μ, η).

Definition 6.2 (Monoid Homomorphism)**.** *A monoid homomorphism $\phi : M \to M$ is a function that commutes with all the structure. So, in terms of compositions, we have*

$$(a\phi) \cdot (b\phi) = (a \cdot b)\phi,$$

and also

$$e\phi = e^{'}.$$

The composition of two monoid homomorphisms is again a monoid homomorphism, and that the identity map is a monoid homomorphism, so altogether: there is a category denoted Mon whose objects are the monoids and whose arrows are the monoid homomorphisms. We write $Mon(X, Y)$ for the set of monoid homomorphisms from X to Y . A monoid homomorphism is called an isomorphism of monoids if there exists a two-sided inverse in Mon.

Definition 6.3 (The Product of 2 Monoids)**.** *If M and $M^{'}$ are two monoids, then the product set $M\ddot{O}M^{'}$ has a canonical monoid structure, namely the one given by component-wise multiplication. That is, the multiplication on $M\ddot{O}M$ is given by:*

$$(MxM^{'})x(MxM^{'}) \to (MxM^{'})$$
$$(x, x^{'}), (y, y^{'}) \mapsto (x \cdot y, x^{'} \cdot y^{'})$$

The unit map $1 \to MxM^{'}$ is simply the product of the two unit maps $1 \to M$ and $1 \to M^{'}$.

Definition 6.4 (Commutative Monoids)**.** *A monoid $(M, \cdot, 1)$ is called commutative if for all elements a, b we have $a \cdot b = b \cdot a$. In terms of arrows and diagrams: a monoid is commutative if the multiplication $\mu : MxM \to M$ is compatible with the twist map.*