Sec 3.2

→ So far, we have defined the derivative of the function $f(x)$, $f'(x)$, using the limit:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{or} \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

→ But this process is quite tedious (Using the definition of limits and limit laws), so we will define the derivative function and find a process for finding it.

Definition of Derivative function: Let f be a function. The derivative function, denoted by f', is the function whose domain consists of those values of x s.t. the following limit exists:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

- A function is said to be differentiable at a if $f'(a)$ exists. More generally, a function f is said to differentiable on S if its differentiable at every point in an open set S.

Key Theorem: Differentiability implies Continuity. If $f(x)$ is differentiable at $a \Rightarrow f$ is continuous at a.

↓ what this tells us.

① If f is not continuous at $x=a \Rightarrow f$ is not differentiable at a.

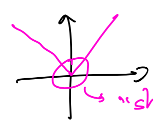
↳ But note that there are continuous functions, which are not differentiable.

↳ Examples: $f(x) = |x|$ at $x=0$.

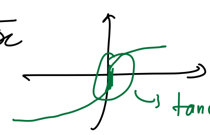
→ In summary, when does a function fail to be differentiable at $x=a$?

① When f is not continuous at $x=a$.

② When the function has a corner (E.g. $f(x) = |x|$,



③ Tangent line is vertical. E.g. $f(x) = \sqrt[3]{x}$



Key differentiation Rules

① If $f(x) = c \Rightarrow f'(x) = 0$.

② When $n \geq 1$, $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

③ Let $f(x), g(x)$ be differentiable functions and k be a constant. Then, the following hold:

(i) $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

(ii) $\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

(iii) $\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)$

④ Product Rule: $\frac{d}{dx}\left(f(x)\,g(x)\right) = f'(x)\,g(x) + f(x)\,g'(x)$.

⑤ Quotient Rule: $\frac{d}{dx}\,\frac{f(x)}{g(x)} = \dfrac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{g(x)^2}$

Name:

Recitation Instructor:

Recitation Time:

Homework #4 is due at 5:00 PM on Feb. 13 in your *recitation's* homework box near Cardwell 120.

1. Using the limit definition of the derivative, find $\frac{d}{dx}x^{-2}$.

$f' \quad (-2)x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$

So, $f(x) = x^{-2}$.

$$f'(x) = \frac{d}{dx}(x^{-2}) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^{-2}-x^{-2}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^2}-\frac{1}{x^2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x^2+2hx+h^2)}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0}\frac{1}{\cancel{h}}\left[\frac{-2\cancel{h}x-\cancel{h}^2}{(x+h)^2x^2}\right]$$

$$= \lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2}$$

quotient rule for limits.

$$= \frac{\lim_{h\to 0}(-2x-h)}{\lim_{h\to 0}x^2(x+h)^2}$$

$$= \frac{-2x}{x^3} = \frac{-2}{x^2}$$

are the same.

2. Using the limit definition of the derivative, find $\frac{d}{dx}\sqrt{x}$.

Same as Qn 2.

3. Using derivative rules, find the following:

A. $\frac{d}{dx}\left(5x^4 + \frac{2}{x^2} + \pi\cdot e^2\right) \underset{\text{Sum rule}}{=} \left(\frac{d}{dx}5x^4\right) + \left(\frac{d}{dx}\frac{2}{x^2}\right) + \left(\frac{d}{dx}\pi e^2\right)$

πe^2 is a constant.

$$= 5(4)x^{4-1} + \frac{d}{dx}2x^{-2} + 0$$

$$= 20x^3 + 2(-2)x^{-2-1} + 0 = 20x^3 - 4x^{-3} \#.$$

B. $\frac{d}{dt}\left(7t^{3/2} - \frac{6}{t^{3/2}}\right)$

↳ Same as (A).

4. When throwing a softball directly upward from a height of 10 ft with an initial velocity of 50 ft/sec, the height of the softball after t seconds is given by $y(t) = -25t^2 + 50t + 10$ (until the ball hits the ground).

A. Using the limit definition of the derivative, find the velocity $y'(t)$.

$$f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$$

$$= \lim_{h\to 0} \frac{[-25(t+h)^2+50(t+h)+10] - [-25t^2+50t+10]}{h}$$

$-25(t+h)^2+50(t+h)+10$
$= -25(t^2+2ht+h^2)+50t+50h+10$
$= -25t^2-50th-25h^2+50t+50h+10$

$$= \lim_{h\to 0} \frac{\cancel{-25t^2} - 50th - 25h^2 \cancel{+50t} + 50h \cancel{+10} \cancel{+25t^2} \cancel{-50t} \cancel{-10}}{h}$$

$$= \lim_{h\to 0} \frac{-50t\cancel{h} - 25h^{\cancel{2}} + 50\cancel{h}}{\cancel{h}} = \lim_{h\to 0} -50t - 25h + 50 = -50t+50$$

$y'(t) = \frac{d}{dt}(-25t^2+50t+10)$
$= \frac{d}{dt}(-25t^2) + \frac{d}{dt}(50t) + \frac{d}{dt}(10)$
$= (-25)(2)t^{2-1} + 50t^{1-1} + 0$
$= -50t+50$
the same!!

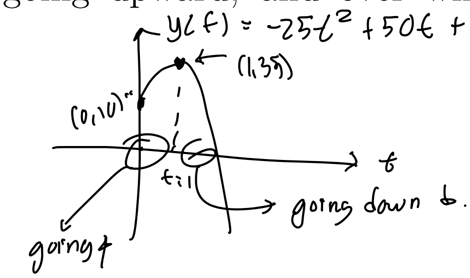
B. When is $y'(t) > 0$, and when is $y'(t) < 0$?

$y'(t) = -50t+50$

$-50t+50>0$
$\Leftrightarrow -50t > -50$
$\Leftrightarrow t < 1$

$-50t+50<0$
$\Leftrightarrow -50t < -50$
$\Leftrightarrow t > 1$

C. Over what time interval is the ball going upward, and over what time interval is the ball going downward?



D. At what time does the ball reach its maximum height?

$\hookrightarrow t = 1.$

E. What is the maximum height reached by the ball?

35.

5. After writing what is considered to be the greatest calculus-inspired rap song of all time, you decide that you want to make money off of your recording by selling the mp3 on your web page. If you price the download at c cents, your song will be purchased $5000 - 10c$ times (for $0 \leq c \leq 500$).

A. Find a formula for $R(c)$, the total revenue generated from sales if the price per download is c cents. $R(c) =$ Total Revenue $=$ (Price/Download) $\times$ total number of purchases

$= c \times (5000 - 10c)$

$= 5000c - 10c^2$.

B. Find a formula for $R'(c)$.

↳ Same as above.

C. Where is $R'(c) > 0$, and where is $R'(c) < 0$.

↳ Same as above.

D. Interpret the meaning of your answer in Part C. What should you charge per download if you want to maximize your total revenue, and how much total revenue will you generate?

↳ $R'(c)$ tells us the rate of change of total revenue at cost c cents. So, if $R'(c) > 0$ at c, this means a small increase in c leads to an increase in $R(c)$. if $R'(c) < 0$ at c, this means a small increase in c leads to a small decrease in revenue. $\therefore$ To maximize total revenue, pick c such that $R'(c) = 0$.

6. Using derivative rules, find the following:

A. $\frac{d}{dt}\left((t^2+t+1)\cdot(4t^2+t+8)\right)$ $= \left[\frac{d}{dt}(t^2+t+1)\right](4t^2+t+8) + (t^2+t+1)\left[\frac{d}{dt}(4t^2+t+8)\right]$

$= (2t+1)(4t^2+t+8) + (t^2+t+1)(8t+1)$

B. $\frac{d}{dx}\left(\frac{3x^2+2x}{x^4+7x^2+3}\right)$ $\longrightarrow$ $\dfrac{\left[\frac{d}{dx}(3x^2+2x)\right](x^4+7x^2+3) - (3x^2+2x)\left[\frac{d}{dx}(x^4+7x^2+3)\right]}{(x^6+7x^2+3)^2}$

$= \dfrac{(6x+2)(x^4+7x^2+3) - (3x^2+2x)\cdot(4x^3+14x)}{(x^4+7x^2+3)^2}$ #

C. $\frac{d}{dx}\left((x+4)(x+5)(x^2+6)\right)$

Combine this 2, then its the same as A.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	4	1	2
2	5	7	2	6
3	12	9	8	-1

7. (4 points each) Using the table above, calculate the following quantities.

A. $a'(2)$ if $a(x) = f(x)g(x)$

$$\frac{d}{dx} a(x) = \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

$$\therefore a'(2) = f'(2)g(2) + f(2)g'(2)$$

$$= 2(7) + 5(6) = 14 + 30 = 44.$$

B. $b'(1)$ if $b(x) = \dfrac{f(x)}{g(x)}$

$$\Rightarrow \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \therefore b'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{g(1)^2}$$

$$= \frac{1(4) - 3(2)}{4^2}$$

$$= \frac{4-6}{16}$$

8. Let $C(x)$ be the cost of producing x cars in a factory.

A. Find a formula for $A(x)$, the average price of producing a car in the factory when x cars are being produced.

$$A(x) = \frac{C(x)}{x}.$$

B. Find $A'(x)$ in terms of x, $C(x)$, and $C'(x)$.

$$A'(x) = \frac{d}{dx}\left(\frac{C(x)}{x}\right) = \frac{\left[\frac{d}{dx}C(x)\right]x - C(x)\left[\frac{d}{dx}x\right]}{x^2}$$

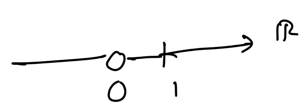
C. When is $A'(x)$ positive, and when is it negative?

$$= \frac{C'(x)\cdot x - C(x)\cdot 1}{x^2}$$

$$= \frac{xC'(x) - C(x)}{x^2}.$$

$$A'(x) > 0 \iff \frac{xC'(x) - C(x)}{x^2} > 0.$$

When is $A'(x) = 0$? Ie when is $\frac{xC'(x) - C(x)}{x^2} = 0$?



D. Explain why your answer in Part C makes sense.

Hint: Marginal cost is the derivative of Cost function.
$\Rightarrow$ Google.

$\iff xC'(x) - C(x) > 0$ and

$x^2 > 0$. (true).

$xC'(x) - C(x) > 0$

$\iff xC'(x) > C(x).$

9. Find $\frac{d}{dx}\left(\frac{(\sqrt{x}+x^2)(x^5-7x+5)}{3x+x^{-2}}\right)$.

↳ Same as above

10. Find $\frac{d^2}{dx^2}(6x^3-37x+8)$.

↳ $\frac{d^{2}}{dx^{2}}$ → So, differentiate the function $6x^3-37x+8$ 2 time.

$$(1)\ \frac{d}{dx}(6x^3-37x+8) = \left(\frac{d}{dx}6x^3\right) - \left(\frac{d}{dx}37x\right) + \left(\frac{d}{dx}8\right)$$

$$= 6\left(\frac{d}{dx}x^3\right) - 37\left(\frac{d}{dx}x\right) + \left(\frac{d}{dx}8\right)$$

$$= 6(3)(x^{3-1}) - 37(1)x^{1-1} + 0 = 18x^2 - 37 \#.$$

$$(2)\ \frac{d}{dx}(18x^2-37) = \left(\frac{d}{dx}18x^2\right) - \left(\frac{d}{dx}37\right)$$

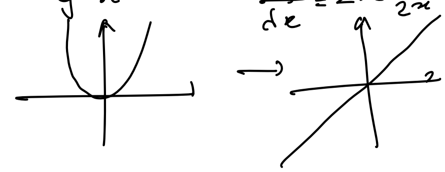
$$= 18\left(\frac{d}{dx}x^2\right) - \left(\frac{d}{dx}37\right)$$

$$= 18(2)x^{2-1} - 0 = 36x \#.$$

Higher Order Derivatives

- The derivative of a function is itself a function. Example:

$y = x^2 \implies \frac{dy}{dx} = 2x$



Another example

Distance function $\xrightarrow{\frac{d}{dt}}$ Velocity function $\xrightarrow{\frac{d}{dt}}$ Acceleration function

Notation

$\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots$

$f'(t), f''(t), f'''(t), \ldots$

- Differentiate f(x) once.

Differentiate f(x) 2 times.

↳ Differentiate f(x) 3 times.

11. If $s(t)$ denotes the position of an object in feet at time t seconds, $s''(t)$ is the acceleration of the object in ft/s^2 at time t seconds. If $s(t) = t^{4/3} - 7t + 4$ ft, find the acceleration at time $t = 8$ s.

↳ Same as above