Notes on Symplectic Manifolds

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1 Introduction

In this notes, I will be introducing the concept of symplectic forms and we will eventually build up to a huge source of examples of symplectic manifolds. These examples, specifically the total space of a cotangent bundle, $T^*(Q)$, of a manifold Q, form the foundation of many of the calculations that will be done in symplectic geometry and in particular, for physics, it is part of the standard examples used in classical mechanics.

But first, I will start with symplectic forms on vector spaces to get a more concrete idea on these objects.

2 Symplectic Forms on Vector Spaces

The entire premise of symplectic forms on vector spaces, and as we will see later in manifolds, is based on a special type of alternating 2-tensor w, which comes with an additional property: $\forall Y \in V, w(X,Y) = 0 \Rightarrow X = 0$. This property is called being non-degenerate.

Definition 2.1 (Non-degenerate 2-tensors)**.** *A 2-tensor w on a finite dimensional vector space V is said to be non-degenerate if $\forall Y \in V, w(X,Y) = 0 \Rightarrow X = 0$.*

Definition 2.2 (Symplectic 2-tensors)**.** *A 2-tensor on a finite dimensional vector space is said to be symplectic if it is alternating and non-degenerate.*

Now let's look at an example of a symplectic tensor. Let V be a $2m$-dimensional vector space. Denote the basis of V be $\{A_1, B_1, A_2, B_2, ..., A_n, B_n\}$ and the dual basis of the dual space V^* to be $\{\alpha^1, \beta^1, \alpha^2, \beta^2, ..., \alpha^n, \beta^n\}$. Let's define the 2-tensor w as $\sum_{k=1}^{n} \alpha^k \wedge \beta^k$. Now, we shall prove that w is symplectic, that is to say it is alternating and non-degenerate.

To show that w is alternating, we just need to see the action of w on the basis elements of V:

$$\begin{aligned} w(A_i, B_j) &= (\sum_{k=1}^{n} \alpha^k \wedge \beta^k)(A_i, B_j) \\ &= \sum_{k=1}^{n} (\alpha^k \wedge \beta^k)(A_i, B_j) \\ &= \sum_{k=1}^{n} \det \begin{pmatrix} \alpha^k(A_i) & \beta^k(A_i) \\ \alpha^k(B_j) & \beta^k(B_j) \end{pmatrix} \\ &= \sum_{k=1}^{n} \alpha^k(A_i)\beta^k(B_j). \end{aligned}$$

The last equivalence is due to the fact that regardless of i, j, $\beta^k(A_i) = \alpha^k(B_j) = 0$. This is because the dual basis α^k and β^k represent the projection of vectors in V onto the repsective A_k and B_k "axes". Similarly, for $w(B_j, A_i)$, we get $-\sum_{k=1}^{n} \alpha^k(A_i)\beta^k(B_j)$. Therefore, $w(A_i, B_j) = -w(B_j, A_i)$, thus proving that w is alternating. Note that, in fact, $w(A_i, B_j) = \delta_{ij}$. Also, using the same technique as above, we can deduce that $w(A_i, A_j) = w(B_i, B_j) = 0$.

To show that w is non-degenerate, we just have prove the statement $\forall Y \in V, w(X, Y) = 0 \Rightarrow X = 0$. So, let X be any arbitrary element in V. Then, X can be expressed to $a^i A_i + b^i B_i$. Suppose that $\forall Y \in V, w(X, Y) = 0$. Then, since the hypothesis applies for all Y in V, we can set $Y = B_i$. Then, we get the following:

$$\begin{aligned} 0 = w(X, B_i) &= w(a^i A_i + b^i B_i, B_i) \\ &= a^i w(A_i, B_i) + b^i w(B_i, B_i) \\ &= a^i. \end{aligned}$$

Similarly, we can see that if we set $Y = A_i$, then we can deduce that $b^i = 0$. Thus, we can see that $X = 0$ and therefore w is non-degenerate.

As w is both alternating and non-degenerate, it is a symplectic form on the $2m$-dimensional vector space V. An important generalisation of this example is the following fact:

Theorem 2.1 (Canonical Form for a Symplectic Tensor)**.** *Let w be a symplectic tensor on an m dimensional vector space V. Then, V has even dimension $m = 2n$, and there exists a basis for V in which w has the form $\sum_{k=1}^{n} \alpha^k \wedge \beta^k$.*

Now, let's turn our attention to symplectic forms on a manifold. A symplectic form on a manifold M is a smooth, closed, non-degenerate 2-form. In other words, a smooth form

w is symplectic if and only if $w : M \to \Lambda^2(M)$ is closed and $\forall p \in M$, w_p is a symplectic tensor. A symplectic manifold is a smooth manifold that is endowed with a symplectic form.

When we look at the definition of a symplectic form, it is not clear that there even are symplectic manifolds, beyond the trivial, flat one $\mathbb{R}^{2n}$. While we can try to construct symplectic manifolds by trying to endow an arbitrary manifold with a closed 2-tensor and then showing the 2-tensor is alternating and non-degenerate, it is quite a difficult thing to do. Fortunately, there is an easier way to construct nice, non-trivial symplectic manifolds. This construction uses the class of smooth manifolds, thus giving us a huge source of examples that we can play with.

Now, we shall show that given any smooth manifold Q, its cotangent bundle $T^*(Q)$ has canonical symplectic structure. Note that $T^*(Q)$ is a smooth manifold in its own right. To show that $T^*(Q)$ is a symplectic manifold, we have to show that $T^*(Q)$ has a smooth 2-form w such that w is closed and $\forall (q, \phi) \in T^*(Q)$, $w_{(q,\phi)}$ is a symplectic tensor.

The 2-form w will be defined as the exterior derivative of the tautological 1-form, τ on $T^*(Q)$:

$$\begin{aligned}
\tau : T^*(Q) &\to T^*(T^*Q) \\
(q, \phi) &\mapsto \tau_{(q,\phi)} \in T^*_{(q,\phi)}(T^*Q) \\
(q, \phi) &\mapsto (\tau_{(q,\phi)} : T_{(q,\phi)}(T^*Q) \to \mathbb{R})
\end{aligned}$$

Let X be a tangent vector in $T_{(q,\phi)}(T^*Q)$. Then

$$\begin{aligned}
\tau_{(q,\phi)}(X) &:= (\pi^*\phi)(X) \\
&= \phi(\pi_* X)
\end{aligned}$$

In the expression above, π is the canonical surjective map from $T^*(Q) \to Q$, that is to say $(q, \phi) \mapsto q$. Then, the pushforward of π is defined as $\pi_* : T_{(q,\phi)}(T^*(Q)) \to T_q(Q)$. The pullback of π is defined as follows: $\pi^* : T^*_q(Q) \to T^*_{(q,\phi)}(T^*Q)$.

To understand why τ is smooth, we require the following lemmas:

Theorem 2.2. *Let $G : M \to N$ be a smooth map, and suppose $f \in C^\infty(N)$ and $w \in T^*(N)$. Then,*

$$\begin{aligned}
G^* df &= d(f \circ G) \\
G^*(fw) &= (f \circ G) G^* w.
\end{aligned}$$

Proof. To prove the first equation, we let $X_p \in T_P M$ be arbitrary, and compute

$$\begin{aligned}(G^* df)_p(X_p) &= (G^*(df_{G(p)})(X_p)) \text{ (by definition of pullback of covector fields)}\\ &= df_{G(p)}(G^* X_p) \text{ (by definition of the pullback, } G^*)\\ &= (G^* X_p)(f) \text{ (by definition of } df)\\ &= X_p(f \circ G) \text{ (by definition of the pullback, } G^*)\\ &= d(f \circ G)_p(X_p) \text{ (by definition of } d(f \circ G)).\end{aligned}$$

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Similarly, for the second equation, we compute

$$\begin{aligned}(G^*(fw))_p &= G^*((fw)_{G(p)}) \text{ (by definition of pullback of covector fields)}\\ &= G^*(f(G(p))_{w_{G(p)}}) \text{ (as smooth covector fields on } M \text{ form a vector space)}\\ &= f(G(p))G^*(w_{G(p)}) \text{ (by linearity of } G^*)\\ &= f(G(p))(G^* w)_p \text{ (definition of pullback } G^*)\\ &= ((f \circ G)G * w)_p \text{ (as smooth covector fields on } M \text{ form a vector space).}\end{aligned}$$

Theorem 2.3. *Suppose $G : M \to N$ is smooth, and let w be a smooth covector field on N. Then, $G^* w$ is a smooth covector field on M.*

Proof. Let $p \in M$ be arbitrary, and choose smooth coordinates (x^i) for M near p and (y^i) for N near $G(p)$. Writing w in coordinates as $w = w_j dy^j$ for smooth functions w_j defined near $G(p)$ and using the lemma above twice, we get the following computation in a neighborhood of p:

$$G^* w = G^*(w_j dy^j) = (w_j \circ G)(G^* dy^j) = (w_j \circ G) d(y^j \circ G).$$

Because the expression above is smooth, it follows that $G^* w$ is smooth.

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This 2 lemmas are important because in the course of the preceding proof, we derived the following formula for the pullback of the covector field with respect to smooth coordinates (x^i) on the domain and (y^j) on the range:

$$G^* w = G^*(w_j dy^j) = (w_j \circ G) d(y^j \circ G) = (w_j \circ G) dG^j,$$

where G^j is the jth component of G in these coordinates. This formula makes the computation of pullbacks in coordinates exceedingly simple. This makes it easy to show that

τ is smooth.

Now, we will formally prove that the tautological 1-form, τ, is smooth, and $w = -d\tau$ is a symplectic form on the total space of T^*Q.

Theorem 2.4. *Let Q be a smooth manifold. The tautological 1-form τ is smooth, and $w = -d\tau$ is a symplectic form on the total space of T^*Q.*

Proof. Let (x^i) be any smooth coordinates on Q, and let (x^i, ϵ_i) denote the corresponding standard coordinates on T^*Q. Recall that the coordinates of $(q, \phi) \in T^*Q$ are defined to be (x^i, ϵ_i), where (x^i) is the coordinate representation of q, and $\epsilon_i dx^i$ is the coordinate representation of ϕ. In terms of these coordinates, the projection $\pi : T^*Q \to Q$ has the coordinate expression $\pi(x, \epsilon) = x$, and therefore the coordinate representation of τ is:

$$\tau_{(x,\epsilon)} = \pi^*(\epsilon_i dx^i) = \epsilon_i dx^i.$$

It follows immediately that τ is smooth, because its component functions are linear.

To show that w is symplectic, we need to show that w is closed and $\forall (q, \phi) \in T^*(Q)$, $w_{(q,\phi)}$ is a symplectic tensor (that is to say, it is alternating and non-degenerate).

By definition, an n-form λ is exact if there exist some $(n-1)$-form γ such that $d\gamma = \lambda$. Thus, w clealy is exact. It is also well-established that every exact n-form is also closed. Thus, w is closed.

Moreover,

$$w = -d\tau = \sum_{i=1}^{n} (dx^i \wedge d\epsilon_i).$$

Under the identification of an open subset of T^*Q with an open subset of $\mathbb{R}^{2n}$ by means of these coordinates, w corresponds to the standard symplectic form on $\mathbb{R}^{2n}$ (with ϵ_i substituted for y^i), which we had seen earlier. Thus, it follows that w is symplectic.

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