# A Model-Informed Deep Learning Algorithm for Solving Inverse Problems

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Joint work with Dinh-Liem Nguyen

# Outline

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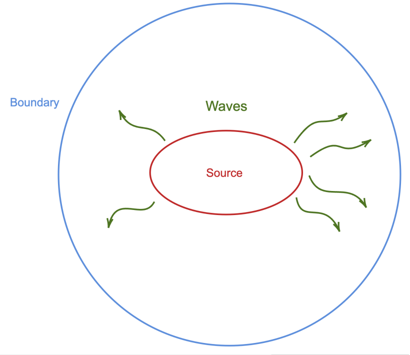
1. Introduction
2. Stable Imaging Function
3. Numerical Results
4. Extension to Inverse Scattering Problems

# Introduction




**Direct source problem:** Given a source that radiates waves, determine the waves.

**Applications:** Radar, non-destructive testing, geophysical exploration, medical imaging, ...

# Inverse source problem

- Let $f \in L^2(\Omega)$, where $\Omega$ is a large disk such that $supp(f) = D \subset \Omega$.

$$\begin{cases} \Delta u + k^2 u = f, & \text{in } \mathbb{R}^2 \\ \frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r^2}\right), & \text{as } r = |x| \to \infty. \end{cases}$$

- Inverse problem: **Given** $u(\cdot, k)|_{\partial\Omega}$ for $k \in [k_1, k_2]$, **determine** $f$.

Uniqueness and stability of solution for the inverse source problem:

- Bao, G., Lin, J., & Triki, F. (2010). A multi-frequency inverse source problem. Journal of Differential Equations, 249, 3443-3465.

**Theorem 1**

*Given a set of real numbers $[k_1, k_2]$, the measurements $u(\cdot, k)|_{\partial\Omega}$, where $k \in [k_1, k_2]$, determine uniquely the source function $f$.*

# Numerical methods

Traditional numerical schemes (far-from-complete list):

- *Bao, G., Lin, J., & Triki, F. (2011). Numerical solution of the inverse source problem for the Helmholtz Equation with multiple frequency data.*
- *Acosta, S., Chow, S., Taylor, J., & Villamizar, V. (2012). On the multi-frequency inverse source problem in heterogeneous media. Inverse Problems, 28.*
- *Eller, M., & Valdivia, N. (2009). Acoustic source identification using multiple frequency information. Inverse Problems, 25, 115005.*
- Bao, G., Lin, J., & Triki, F. (2010). A multi-frequency inverse source problem. Journal of Differential Equations, 249, 3443-3465.
- Kress, R., & Rundell, W. (2013). Reconstruction of extended sources for the Helmholtz equation. Inverse Problems, 29.
- Karamehmedovi'c, M., Kirkeby, A., & Knudsen, K. (2018). Stable source reconstruction from a finite number of measurements in the multi-frequency inverse source problem. Inverse Problems, 34.
- Eibert, T.F. (2023). Multiple-Frequency Preconditioned Iterative Inverse Source Solutions. IEEE Transactions on Microwave Theory and Techniques, 71, 2842-2853.

Severely illposed $\implies$ numerical schemes require regularization strategies

# Deep learning methods

Deep learning-based algorithms (far-from-complete list):

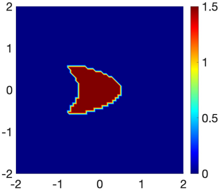
- *Dong, Y., Sadiq, K., Scherzer, O., & Schotland, J.C. (2024). Computational inverse scattering with internal sources: a reproducing kernel Hilbert space approach. Physical review. E, 110 6-2, 065302 .*
- *Meng, S., & Zhang, B. (2024). A Kernel Machine Learning for Inverse Source and Scattering Problems. SIAM J. Numer. Anal., 62, 1443-1464.*
- *Zhang, H., & Liu, J. (2023). Solving an inverse source problem by deep neural network method with convergence and error analysis. Inverse Problems, 39.*

Solve inverse source problems with "internal" data!

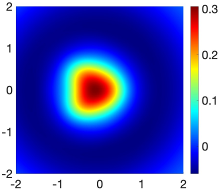
# Proposed Algorithm

We developed a 2-step **unsupervised** deep learning algorithm to solve the inverse source problem:

- Step 1: Convert the boundary data into an imaging function $\mathcal{I}(z)$ which encodes geometrical information about $f$.
- Step 2: Use $\mathcal{I}(z)$ as input for training. A model equation for $\mathcal{I}(z)$ and $f$ is employed to train a neural network which predicts $f$.




(a) Original source




(b) Data from $\mathcal{I}$

# Outline

# Stable Imaging Function

- For each $k \in [k_1, k_2]$, for $z \in \mathbb{R}^2$, define the function

$$\mathcal{I}(z,k) := \int_{\partial\Omega} u(x,k)\overline{\Phi_k(x,z)}\, dS(x),$$

where $\Phi_k(x,z)$ denotes the Green's function.

We assume the noisy data $u_\delta$ satisfies

$$\|u - u_\delta\|_{L^2(\partial\Omega\times\mathbb{S})} \le \delta \|u\|_{L^2(\partial\Omega\times\mathbb{S})}.$$

**Theorem 2**

*Let $\mathcal{I}_\delta(z)$ the imaging function with noisy data $u_\delta$. Then*

$$|\mathcal{I}(z) - \mathcal{I}_\delta(z)| \le C\delta, \quad \text{for all } z \in \mathbb{R}^2,$$

*where $C$ is a positive constant which is independent of $z$ and $\delta$.*

# Significance of the model equation

## Theorem 3

*The imaging function* $\mathcal{I}(z)$ *satisfies*

$$\mathcal{I}(z,k) = \frac{1}{4k}\int_{\Omega} J_0(k|y-z|)\, f(y)\, dy$$

Proof: Using the volume potential and the Helmholtz-Kirchhoff identity.

Significance:

- A simple and nice connection between $\mathcal{I}(z)$ and $f$.
- This equation will be used as a model equation in training our neural network with data $\mathcal{I}(z)$.

# Model Informed Neural Network

Key Ideas:

- $\mathcal{I}(z)$ can be obtained robustly and inexpensively from (noisy) scattering data.
- Use input data $\mathcal{I}(z)$ and the model equation to train neural network $f_\Theta$ to predict $f$.

# Training Process

Ansatz Space/Hypothesis space $\mathcal{H}$ : Space of Neural Networks with the following configuration:

- Depth of Neural Network: 4
- Width of Neural Network: 64
- Activation Functions: Hyperbolic Tangent Function

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With the chosen space $\mathcal{H}$ in place, our physics-based loss functional $\mathcal{L}[\cdot]$ is:

$$\mathcal{L} : \mathcal{H} \to \mathbb{R}$$

$$f_\theta \mapsto \frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{N}\sum_{j=1}^{N}\left[\mathcal{I}(z_j, k_i) - \frac{1}{4k_i}\int_\Omega J_0(k_i|z_j - y|)f_\theta(y)dy\right]\right).$$

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Minimization Problem:

$$\min_{w_\theta \in \mathcal{H}} \frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{N}\sum_{j=1}^{N}\left[\mathcal{I}(z_j, k_i) - \frac{1}{4k_i}\int_\Omega J_0(k_i|z_j - y|) w_\theta(y)dy\right]\right)$$

# Visual Representation

Let $z = (x, y)^\top \in \mathbb{R}^2$.

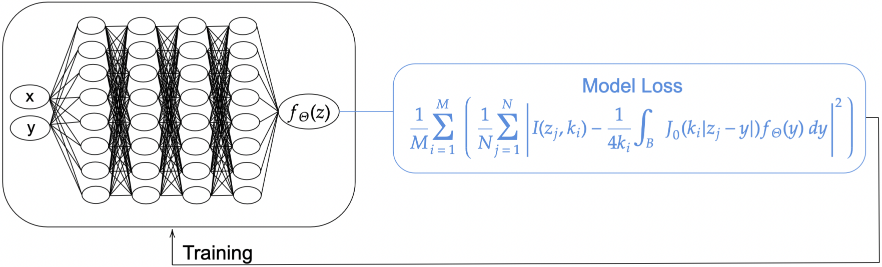
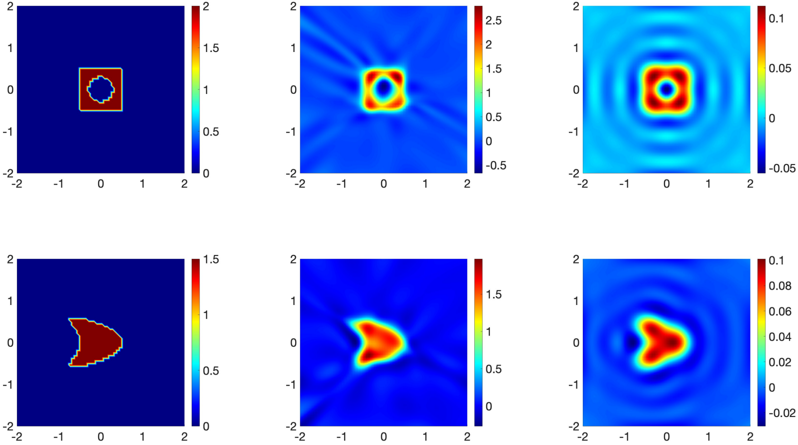



Figure 2: Visual representation of the model-informed neural network

# Outline

# Numerical results

Reconstruction with 20% artifical noise introduced to the boundary data:




(a) Original Source (b) Predicted (c) Least Square

# Review

We have developed an unsupervised, model-informed deep learning algorithm to solve inverse source problems

- In step 1, regularization is done automatically $\Longrightarrow$ no need for additional regularization
- The algorithm incorporates a simple model integral equation into the deep learning process instead of PDE models, eliminating the need for automatic differentiation $\Longrightarrow$ computationally cheaper
- The method is highly robust against noisy data, ensuring stability and accuracy in practical applications.

# Outline

# Extension - Inverse Scattering Problems

Can be extended to solve inverse scattering problems too

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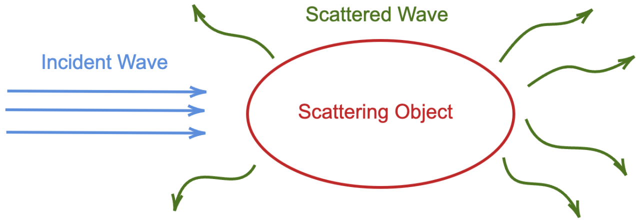



**Inverse scattering problem:** Determine the *scattering object* from boundary measurements of scattered wave (for several incident waves).

**Applications:** Radar, non-destructive testing, geophysical exploration, medical imaging, ...

# Mathematical Formulation

- Let $\eta : \mathbb{R}^2 \to \mathbb{R}$ be bounded function satisfying $\eta = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Let $\Omega$ be large disk such that $D \subset \Omega$. Consider incident waves

$$u^{in}(x, d) = e^{ikd\cdot x}$$

where $d \in \mathbb{S}^1 := \{x \in \mathbb{R}^2 : |x| = 1\}$.

- Consider the following model problem

$$\begin{cases} \Delta u + k^2(1 + \eta(x))u = 0, & \text{in } \mathbb{R}^2 \\ u = u^{in} + u^{sc}, & \\ \lim\limits_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial u^{sc}}{\partial r} - iku^{sc}\right) = 0, & r = |x|. \end{cases}$$

- **Inverse Problem:** **Given** $u^{sc}(\cdot, d)|_{\partial\Omega}$ for all $d \in \mathbb{S}^1$, **determine** $\eta$.

# Deep Learning for inverse scattering

Supervised learning-based algorithms (far-from-complete list)

- *Wei, Z., & Chen, X. (2019). Deep-Learning Schemes for Full-Wave Nonlinear Inverse Scattering Problems. IEEE Transactions on Geoscience and Remote Sensing, 57, 1849-1860.*
- *Khoo, Y., & Ying, L. (2018). SwitchNet: a neural network model for forward and inverse scattering problems. ArXiv, abs/1810.09675.*
- *Sanghvi, Y., Kalepu, Y., & Khankhoje, U.K. (2020). Embedding Deep Learning in Inverse Scattering Problems. IEEE Transactions on Computational Imaging, 6, 46-56.*
- *Chen, X., Wei, Z., Li, M., & Rocca, P. (2020). A Review of Deep Learning Approaches for Inverse Scattering Problems (Invited Review). Progress In Electromagnetics Research.*
- *Gao, Y., Liu, H., Wang, X., & Zhang, K. (2021). On an artificial neural network for inverse scattering problems. J. Comput. Phys., 448, 110771.*
- *Zhou, M., Han, J., Rachh, M., Borges, C. (2022). A Neural Network Warm-Start Approach for the Inverse Acoustic Obstacle Scattering Problem. ArXiv, abs/2212.08736.*
- ...

Reconstruct unknown scattering objects that are similar or closely related to known training datasets!

# Deep Learning for inverse scattering

Unsupervised learning: Physics Informed Neural Networks (PINNs)


- *Raissi, M., Perdikaris, P., & Karniadakis, G.E. (2019). Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. J. Comput. Phys., 378, 686-707.*
- *Chen, Y., Lu, L., Karniadakis, G.E., & Dal Negro, L. (2019). Physics-informed neural networks for inverse problems in nano-optics and metamaterials. Optics express, 28 8, 11618-11633 .*
- *Chen, Y., & Dal Negro, L. (2021). Physics-informed neural networks for imaging and parameter retrieval of photonic nanostructures from near-field data. APL Photonics.*
- *Pokkunuru, A., Rooshenas, P., Strauss, T., Abhishek, A., & Khan, T.R. (2023). Improved Training of Physics-Informed Neural Networks Using Energy-Based Priors: a Study on Electrical Impedance Tomography. International Conference on Learning Representations.*


Study inverse design problems with “internal” data!

# Imaging Function

- For $z \in \mathbb{R}^2$, define the function

$$\mathcal{I}(z) := \int_S \int_{\partial\Omega} u^{sc}(x,d)\overline{\Phi(x,z)}\,ds(x)\,\Phi^\infty(d,z)\,ds(d),$$

  where $\Phi(x,z)$ and $\Phi^\infty(d,z)$ denote the Green's function and its scattering amplitude respectively.

With the Helmholtz-Kirchhoff Identity, Funk-Hecke Formula and Born approximation, we can derive

$$\mathcal{I}(z) - \frac{k\pi}{2}\int_D J_0^2(k|y-z|)\,\eta(y)\,dy = 0.$$

- A simple and nice connection between $\mathcal{I}(z)$ and $\eta$.
- This equation will be used as a (simplified) model equation in training our neural network with data $\mathcal{I}(z)$.

# Stable Imaging Function

We assume the noisy data $u_\delta^{sc}$ satisfies

$$\|u^{sc} - u_\delta^{sc}\|_{L^2(\partial\Omega\times\mathbb{S})} \leq \delta\|u^{sc}\|_{L^2(\partial\Omega\times\mathbb{S})}.$$

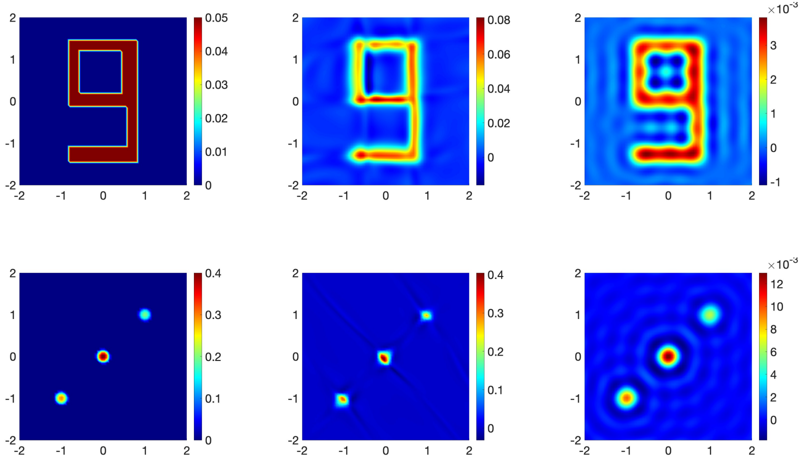
**Theorem 4**

*Let $I_\delta(z)$ the imaging function with noisy data $u_\delta^{sc}$. Then*

$$|\mathcal{I}(z) - \mathcal{I}_\delta(z)| \leq C\delta, \quad \text{for all } z \in \mathbb{R}^2,$$

*where $C$ is a positive constant which is independent of $z$ and $\delta$.*

# Some numerical results




(a) Original Scatterer (b) Predicted (c) Least Square

# Review

We have developed an unsupervised, model-informed deep learning algorithm to solve inverse scattering problems

- Unsupervised $\implies$ greater generalisability
- Similar to the case of the algorithm for the inverse source problem, the algorithm is computationally cheap
- Again, the method is highly robust against noisy data, ensuring stability and accuracy in practical applications.

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Thank you for listening!