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Week 1

~~Sy~~ Grading

- Exam 1: 15% → 15^{th} Sept
- Exam 2: 15% → 13^{th} Oct
- Exam 3: 15% → 10^{th} Nov
- Recitation Grade: 5%
- Written homework: 15% (Will be posted on Canvas. Each recitation has its own box near Cardwell 120. Write your name and your instructor's name at the top.)
- Online Homework: 10% → (Access through Canvas. Multiple submissions is allowed before due date).
- Finals: 25% → 14^{th} Dec.

- No calculators, books or formula sheet on the exams.

Section 1.1

⑦ $f(x) = 5x - 2$.

- Remember that a function can be visualized as an input/output device.

In this case, the function f takes in x and gives out $5x-2$ as an output.

Look at the variable x as a placeholder. So, we have.

a) $f(0) = 5(0) - 2 = -2$

b) $f(1) = 5(1) - 2 = 3$.

c) $f(3) = 5(3) - 2 = 13$.

d) $f(-x) = 5(-x) - 2 = -5x - 2$

e) $f(a) = 5(a) - 2 = 5a - 2$

f) $f(a+h) = 5(a+h) - 2$
$= 5a + 5h - 2$

⑯ Let $f(x) = \dfrac{3}{x^2+4}$

- Recall the definition of domain & range of the function.

Domain → Set of inputs.
Range → Set of outputs.

- To determine the set of inputs, we should look for values which make the function invalid.

So, we know that we can't have a number divided by 0. That is to say the following expression is invalid: $\frac{a}{0}$.

Therefore, we know that $x^2 + 4 \neq 0$.

So, we need to find out values for which $x^2+4=0$ and exclude this from ~~all~~ the sets of inputs.

However, Note that for all x, $x^2 \geq 0$.

$\therefore x^2 + 4 \geq 4 > 0$.

$\therefore$ Domain $= \mathbb{R}$.

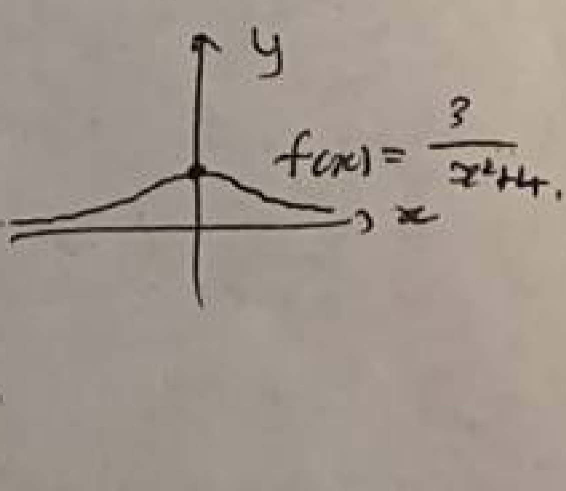
→ To find the range, it takes a little more work. Note that in the function, we are dividing 3 by x^2+4, for $\forall x \in \mathbb{R}$. So, as $x \to \infty$, x^2+4 increases. So, the denominator gets bigger and bigger. Therefore, the denominator gets the smallest when $x = 0$.

$\therefore$ maximum value of range is $\dfrac{3}{0^2+4} = \dfrac{3}{4} = f(0)$.

As $x \to \infty$, $x^2 + 4 \to \infty$.

$\therefore \dfrac{3}{x^2+4} \to 0$.

Thus, the graph looks like



Week 1 | (82) $6x-5y+15=0$.
To get the slope, and y-intercept b, we

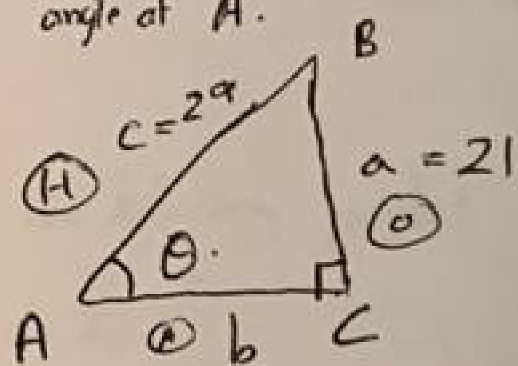
(118) Convert the angle in radians to degrees: $\frac{\pi}{2}$ rad.

2π rad $= 360°$

π rad $= 180°$

$\frac{\pi}{2}$ rad $= \frac{180°}{2} = 90°$

(130) Consider the $\triangle ABC$, with $\llcorner$ at C.
① Find the missing side of the $\triangle$.
② Find the 6 trigonometric values for the angle at A.



① By pythagoras thm,
$c^2 = a^2 + b^2$
$29^2 = 21^2 + b^2$
$\therefore b^2 = 29^2 - 21^2$
$= 400.$
$\therefore b = \pm\sqrt{400}$
$= \pm 20$
(rej -20 as length can't be negative).

② Let angle at a be θ.

$\sin\theta = \frac{O}{H} = \frac{21}{29}$

$\cos\theta = \frac{A}{H} = \frac{20}{29}$

$\tan\theta = \frac{O}{A} = \frac{21}{20}$

$\csc\theta = \frac{H}{O} = \frac{29}{21}$

$\sec\theta = \frac{H}{A} = \frac{29}{20}$

$\cot\theta = \frac{A}{O} = \frac{20}{21}$

(140) $\sec(x)\sin(x)\cot(x)$ Week 1

$= \frac{1}{\cos(x)} \cdot \sin(x) \cdot \cot(x)$

$= \frac{1}{\cos(x)} \cdot \sin x \cdot \frac{1}{\tan(x)}$

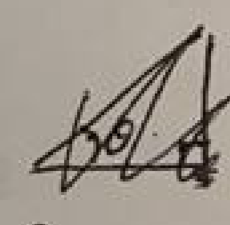
$* \frac{1}{\tan(x)} = 1 \div \tan(x)$
$= 1 \div \frac{\sin(x)}{\cos(x)}$
$= \frac{\cos(x)}{\sin(x)}$

$\therefore = \frac{1}{\cancel{\cos(x)}} \cdot \cancel{\sin(x)} \cdot \frac{\cancel{\cos(x)}}{\cancel{\sin(x)}}$

$= 1$

(156) $1 + \cos(\theta) = \frac{1}{2}$

$\Rightarrow \cos(\theta) = -\frac{1}{2}.$



$\theta = \cos^{-1}\left(-\frac{1}{2}\right).$

Sec 2.1

④ For the following exercise, points $P(1,1)$ & $Q(x,y)$ are on the graph of the function $f(x) = x^3$.

x	y	$Q(x,y)$	m_{sec}
1.1	1.331	(1.1, 1.331)	$\frac{1.331-1}{1.1-1} = 3.31$
1.01	1.030301	(1.01, 1.030301)	$\frac{1.030301-1}{1.01-1} = 3.0301$
1.001	1.003003001	(1.001, 1.003003001)	$\frac{1.003003001-1}{1.003-1} = 3.003$
1.0001	1.000300030001	(1.0001, 1.00030003001)	$\frac{1.00030003001-1}{1.0001-1} = 3.00030001$

⑤ 3. ⑥ Eqn of line $f(x) - y_1 = m(x - x_1)$
$(x_1, y_1) = (1,1)$
$m = 3.$

(43) Find $(f\circ g)(x)$ and $(g\circ f)(x)$. Week 1

$f(x) = 3x,\ g(x) = x+5.$

By definition of $(f\circ g)(x)$, given the input x, we send x through function g, getting the output $g(x)$ and then, let f act on $g(x)$, getting $f(g(x))$.

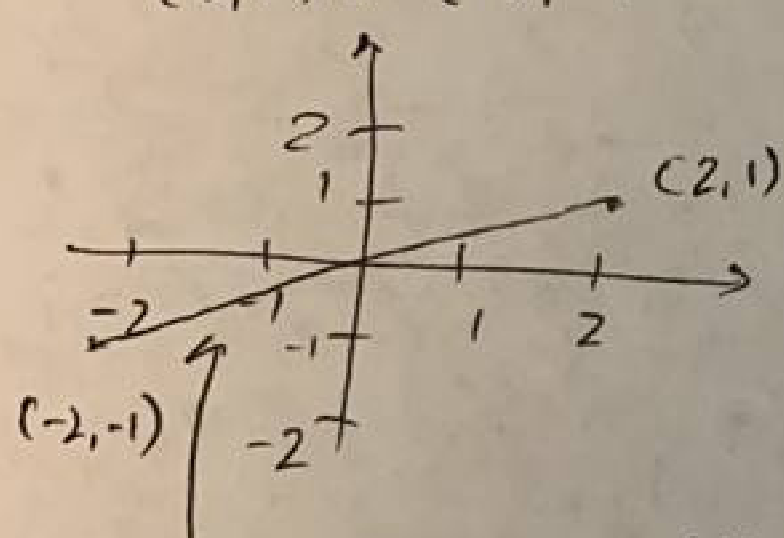
$$x \longmapsto \boxed{g} \longrightarrow g(x) \longmapsto \boxed{f} \longrightarrow f(g(x)) = (f\circ g)(x).$$

$\therefore$ We have
$$(f\circ g)(x) = f(g(x)) = f(x+5) = 3(x+5) = 3x+15.$$

Section 1.2

(71) We are given the following points:

$(2,1)$ & $(-2,-1)$



We want to find the eqn of the line.

To find the slope: recall the definition,

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \underset{x_1\ \ y_1}{(2,1)} \quad \underset{x_2\ \ y_2}{(-2,-1)}$$

$$= \frac{(-1)-(1)}{-2-2} = \frac{-2}{-4} = 1/2.$$

Slope intercept of a linear function:

$$f(x) = \underset{\text{slope}}{m}x + b \quad \hookrightarrow \text{y-intercept.}$$

Here, we can't find b.

So, we use the point-slope eqn: ~~for~~ ~~$y - y_1$~~ $f(x) - y_1 = m(x - x_1)$

$\Rightarrow f(x) - 1 = \frac{1}{2}(x-2)$

$\Rightarrow f(x) = \frac{1}{2}x - 1 + 1 \Rightarrow f(x) = \frac{1}{2}x$!

(82) $6x - 5y + 15 = 0.$

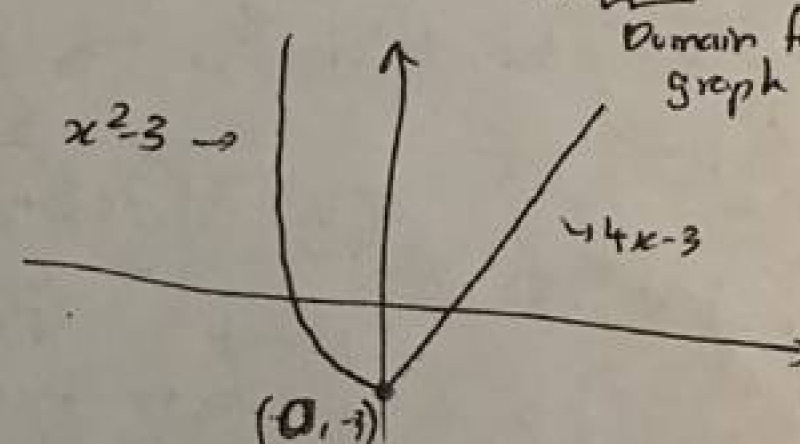
To get the slope, and y-intercept b, we need to express the line in the form of

$$y = mx + b.$$

So, let's do that.

$$6x - 5y + 15 = 0$$
$$6x - 5y + 15 + 5y = 0 + 5y$$
$$6x - 15 = 5y$$
$$\Rightarrow 5y = 6x - 15$$
$$y = \frac{6}{5}x - 3$$
$$y = \underset{\text{Gradient } m.}{\frac{6}{5}}x + \underbrace{(-3)}_{\text{y-intercept}}$$

(95) $f(x) = \begin{cases} x^2 - 3, & x < 0 \leftarrow \text{Domain for the other piece.} \\ 4x - 3, & \underbrace{x \geq 0}_{\text{Domain for 1 piece of the graph}} \end{cases}$



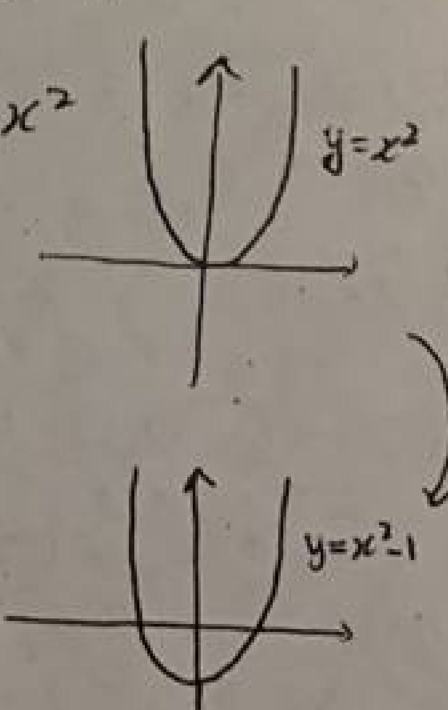
we check these 2 separately.

$f(-4) = (-4)^2 - 3 = 16 - 3 = 13.$

$f(0) = (0)^2 - 3 = -3.$

$f(2) = 2^2 - 3 = 1.$

(88) $f(x) = x^2$



$g(x) = x^2 - 1 = \underbrace{f(x) - 1}_{}$

vertical shift down 1 unit.

Sec 2.2

Week 2

- Consider the function $f(x) = (1+x)^{1/x}$.

(32) Make a table showing the values of f for

$x = -0.01, -0.001, -0.0001, -0.00001$
and for

$x = 0.01, 0.001, 0.0001, 0.00001$

x	f(x)	x	f(x)
-0.01	2.731999	0.01	2.70481
-0.001	2.719642	0.001	2.716924
-0.0001	2.718418	0.0001	2.718146
-0.00001	2.718295	0.00001	2.718263

(33) $\lim_{x\to 0} f(x) = 2.72.$

(45) Set up a table of values and round up to 8 significance digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

$$\lim_{\alpha\to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$$

α	$\frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$
0.1	10
0.01	100
0.001	1000
0.0001	10000

Sec 2.3

(38) $\lim_{x\to -2} (4x^2-1) = \lim_{x\to -2} 4x^2 + \lim_{x\to -2} (-1)$

$= 4\lim_{x\to -2} x^2 + \lim_{x\to -2} (-1)$

$= 4(-2)^2 + (-1)$

$= 16-1$

$= 15.$

(94) $\lim_{x\to 2} \frac{x-2}{x^2-2x}$

$= \frac{\lim_{x\to 2}(x-2)}{\lim_{x\to 2}(x^2-2x)}$

$= \frac{\lim_{x\to 2} x + \lim_{x\to 2}(-2)}{\lim_{x\to 2} x^2 + \lim_{x\to 2}(-2x)}$

$= \frac{\lim_{x\to 2} x + \lim_{x\to 2}(-2)}{\lim_{x\to 2} x^2 + (-2)\lim_{x\to 2} x}$

$= \frac{2-2}{4-4} = \frac{0}{0}$

∴ it is indeterminate.

∴ Since $\frac{x-2}{x^2-2x} = \frac{x-2}{x(x-2)} = \frac{1}{x}.$

∴ $\lim_{x\to 2} \frac{x-2}{x^2-2x} = \lim_{x\to 2} \frac{1}{x} = \frac{\lim_{x\to 2} 1}{\lim_{x\to 2} x} = \frac{1}{2}.$

Week 2

(101) $$\lim_{x \to 1/2} \frac{2x^2+3x-2}{2x-1}$$

$$= \frac{\lim_{x \to 1/2} (2x^2+3x-2)}{\lim_{x \to 1/2} (2x-1)}$$

$$= \frac{2\lim_{x \to 1/2} x^2 + 3\lim_{x \to 1/2} x + \lim_{x \to 1/2} (-2)}{2\lim_{x \to 1/2} x + \lim_{x \to 1/2} (-1)}$$

$$= \frac{\frac{2}{4} + \frac{3}{2} - 2}{1-1} = \frac{0}{0}.$$

$\therefore$ We will simplify $$\frac{2x^2+3x-2}{(2x-1)} = \frac{(2x-1)(x+2)}{(2x-1)}$$

$$= x+2.$$

$$\therefore \lim_{x \to 1/2} \frac{2x^2+3x-2}{2x-1} = \lim_{x \to 1/2} (x+2)$$

$$= \lim_{x \to 1/2} x + \lim_{x \to 1/2} 2$$

$$= \frac{1}{2} + 2 = 2.5.$$

(104) $$\lim_{x \to -2^+} \frac{2x^2+7x-4}{x^2+x-2} = \frac{2\lim_{x \to -2^+} x^2 + 7\lim_{x \to -2^+} x + \lim_{x \to -2^+} (-4)}{\lim_{x \to -2^+} x^2 + \lim_{x \to -2^+} x + \lim_{x \to -2^+} (-2)}$$

$$= \frac{2(-2)^2 + 7(-2) + (-4)}{(-2)^2 + (-2) - 2} = \frac{8-14-4}{0} = \frac{-10}{0}.$$

$$\lim_{x \to -2^+} \frac{(2x-1)(x+4)}{(x+2)(x-1)} = \underbrace{\left(\lim_{x \to -2^+} (2x-1)\right)}_{\to -5} \underbrace{\left(\lim_{x \to -2^+} x+4\right)}_{-2} \underbrace{\left(\lim_{x \to -2^+} \frac{1}{x+2}\right)}_{\to \infty} \cdot \underbrace{\left(\lim_{x \to -2^+} \frac{1}{x-1}\right)}_{1}$$

$\therefore$ Limit $= \infty$.

• Definition of Derivative of tangent line: Let $f(x)$ be a function, defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'(a)$, is defined by,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \quad \text{OR (alternatively)} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h)-a}.$$

Week 3

Sec 3.1, Qn 14

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h)-a}$$

$$= \lim_{h \to 0} \frac{1-(a+h)-(a+h)^2-(1-a-a^2)}{h}$$

$$= \lim_{h \to 0} \frac{1-a-h-a^2-2ah-h^2-1+a+a^2}{h}$$

$$= \lim_{h \to 0} \frac{-h-2ah-h^2}{h}$$

$$= \lim_{h \to 0} -1-2a-h$$

$$= -1-2a.$$

Sec 3.1, Qn 24

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{3x^2-x+2-3a^2+a-2}{x-a}$$

$$= \lim_{x \to a} \frac{3x^2-3a^2-(x-a)}{x-a}$$

$$= \lim_{x \to a} \frac{3(x-a)(x+a)-(x-a)}{x-a}$$

$$= \lim_{x \to a} 3(x+a)-1$$

$$= 6a-1.$$

Sec 3.1 Qn 37

(a) $$V_{ave} = \frac{s(t)-s(a)}{t-a} = \frac{s(2+h)-s(2)}{h}$$

$$= \frac{2(2+h)^3+3-2(2)^3-3}{h}$$

$$= \frac{2(8+12+6h^2+h^3)+3-16-3}{h}$$

$$= \frac{16+32h+12h^2+2h^3-16}{h} = 32+12h+2h^2$$

(b)

$h=0.1$	$s(0.1)=33.22$
$h=0.01$	$s(0.01)=32.12$
$h=0.001$	$s(0.001)=32.012$
$h=0.0001$	$s(0.0001) = 32.001$

(c) 32.0

Week 3

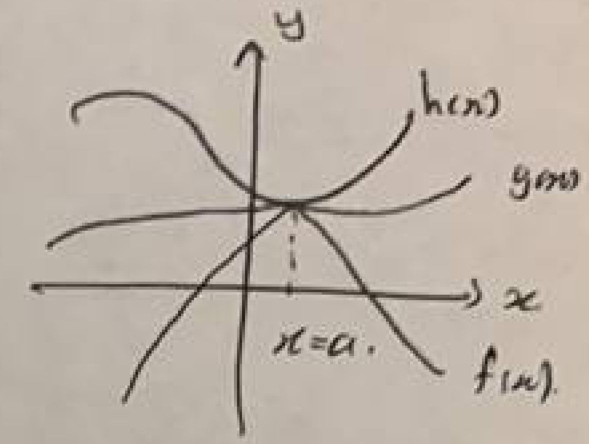
Wk 3

- Squeeze thm: let $f(x), g(x), h(x)$ be defined for all $x \neq a$, over an open interval containing a.

If $f(x) \leq g(x) \leq h(x)$, $\forall x \neq a$ in an open interval containing a and

& $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$, where L is a real number,

then $\lim_{x \to a} g(x) = L$.



Sec 2.3, Qn 128

- We need to find the upper and lower function.

$g(x) = 0, \forall x$

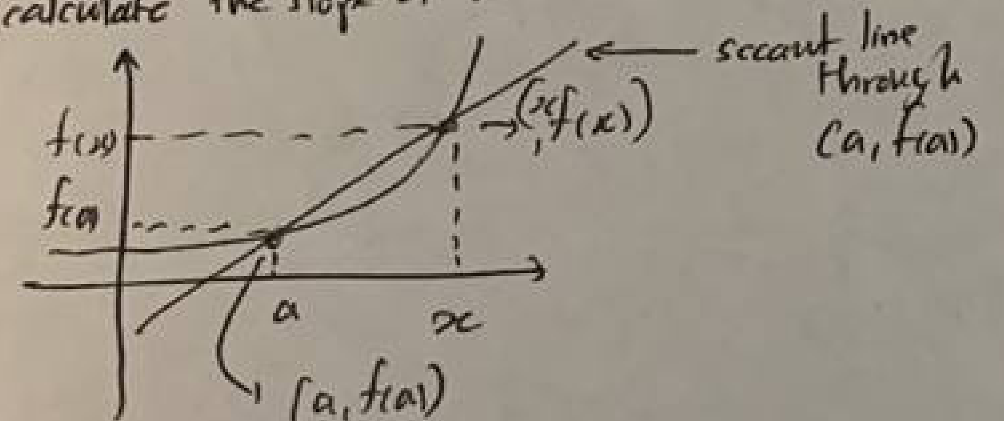
$h(x) = x^2, \forall x.$

$\Rightarrow \lim_{x \to 0} g(x) = \lim_{x \to 0} 0 = 0$

$\lim_{x \to 0} h(x) = \lim_{x \to 0} x^2 = 0$ } as x^2 is a ctn function.

Using squeeze thm. $\lim_{x \to 0} f(x) = 0$.

- Remember to calculate the slope of the secant line.

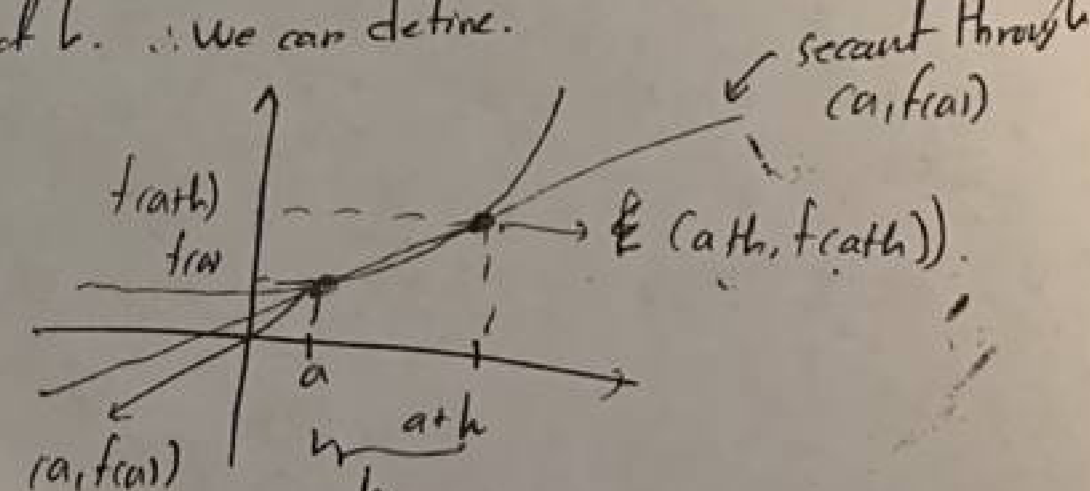


$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$

Note that x can also be seen as $a+h$, for some value of h. ∴ we can define.

$$m_{sec} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$= \frac{f(a+h) - f(a)}{h}.$$



- the tangent line represents the limit of the secant lines.

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{OR (alternatively)} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}.$$

Sec 3.2, Qn 57

Week 3

- Defi of derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{4(x+h)^2 - 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2}{h}$$

$$= \lim_{h \to 0} 8x + 4h = 8x.$$

Sec 3.9, Qn 69

$$\lim_{h \to 0} \frac{[3(2+h)^2 + 2] - 14}{h}$$

→ The h tells us that the relevant formula is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

$$\therefore f(a+h) = 3(2+h)^2 + 2$$

$$f(a) = 14.$$

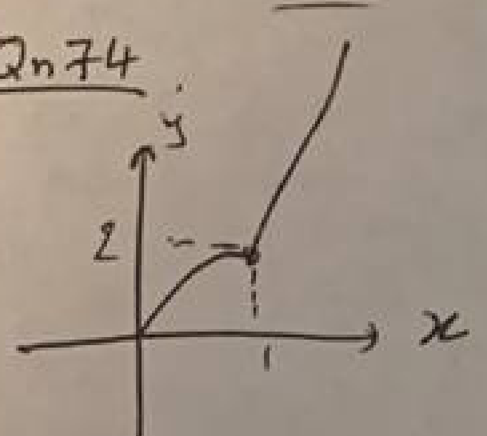
$$\Rightarrow f(x) = 3x^2 + 2.$$

$$3x^2 + 2 = 14 \Leftrightarrow 3x^2 = 12$$

$$\Leftrightarrow x^2 = 4$$

$$\Leftrightarrow x = \pm 2.$$

Sec 3.2 Qn 74



- In order for $f(x)$ to be differentiable at $x=1$, the limit

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ must exist.

$$\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

ie $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$.

However, $LHS = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{2\sqrt{1+h} - 2\sqrt{1}}{h} = \lim_{h \to 0} 2\left(\frac{\sqrt{1+h} - 1}{h}\right).$

$$= \lim_{h \to 0} 2\left(\frac{(1+h) - 1}{h\sqrt{1+h} + h}\right) = \lim_{h \to 0} \frac{h}{h\sqrt{1+h} + h} = \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = 1/2.$$

However, $\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3(1+h) - 1 - 3 + 1}{h} = \lim_{h \to 0} \frac{3 + 3h - 3}{h} = 3$. $\therefore$ limit doesn't exist.

Sec 3.3 Qn 107

Week 3

$$\frac{d}{dx} f(x) = \frac{d}{dx}(5x^3 - x + 1) = \frac{d}{dx} 5x^3 + \frac{d}{dx}(-x) + \frac{d}{dx}(1)$$

$$= 5\frac{d}{dx}x^3 + (-1)\frac{d}{dx}x + \frac{d}{dx}(1)$$

$$= 15x^2 - 1.$$

Sec 3.3 Qn 122

$$\frac{d}{dx} h(x) = \left(\frac{d}{dx} x^3\right) h(x) + x^3 \frac{d}{dx} f(x)$$

$$= 3x^2 h(x) + x^3 f'(x).$$

Differentiation rules

Constant rule: Let $f(x) = C$. Then, $f'(x) = 0$. ie $\frac{d}{dx}(c) = 0$.

Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum and difference rule: $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$.

Constant multiple rule: $\frac{d}{dx} k(f(x)) = k\frac{d}{dx} f(x)$.

Product rule: $\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x))\, g(x) + f(x)\frac{d}{dx} g(x)$.

Quotient rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx} f(x)\right) \cdot g(x) - f(x) \cdot \frac{d}{dx}(g(x))}{(g(x))^2}$.

Week 4

Differentiation Rules

① Let c be a constant. If $f(x) = c \Rightarrow f'(c) = 0$. ie $\frac{d}{dx}(c) = 0$.

② Power Rule: $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$, ie $\frac{d}{dx}x^n = nx^{n-1}$

③ Sum, Difference and Constant Multiple Rules:

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \left(\frac{d}{dx}f(x)\right) \pm \left(\frac{d}{dx}g(x)\right)$$

$$\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x).$$

④ Product Rule: $\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right) \cdot g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$

⑤ Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx}f(x)\right) \cdot g(x) - \left(\frac{d}{dx}g(x)\right) \cdot f(x)}{g(x)^2}$

Sec 3.3 Qn 117: Find $f'(x)$.

$$f(x) = \frac{x+9}{x^2-7x+1}$$

$$f(x) = (x+9)(x^2-7x+1)^{-1}$$

$$\therefore \frac{d}{dx}f(x) = \frac{d}{dx}\left[(x+9)(x^2-7x+1)^{-1}\right]$$

$$= \left[\frac{d}{dx}(x+9)\right](x^2-7x+1)^{-1} + (x+9)\frac{d}{dx}(x^2-7x+1)^{-1}$$

$$= \left[\frac{d}{dx}x + \frac{d}{dx}9\right](x^2-7x+1)^{-1} + (x-9)\left[(-1)(x^2-7x+1)^{-2}\left\{\frac{d}{dx}(x^2-7x+1)\right\}\right].$$

$$= (1+0)(x^2-7x+1)^{-1} + (x-9)\left(\frac{-1}{(x^2-7x+1)^2}\left\{\frac{d}{dx}x^2 + \frac{d}{dx}(-7x) + \frac{d}{dx}(1)\right\}\right).$$

$$= \frac{1}{x^2-7x+1} + \frac{-(x-9)}{(x^2-7x+1)^2}(2x-7)$$

$$= \frac{1}{x^2-7x+1} - \frac{(x-9)(2x-7)}{(x^2-7x+1)^2}$$

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Week 4

• The instanteous rate of change at a is its derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

• Let $s(t)$ be a function giving the position of an object at time t.

→ Velocity a time t: $v(t) = s'(t)$.

→ Speed: $|v(t)|$

→ accerelation $a(t) = v'(t) = s''(t)$

Ex3.4, Qn 15)

The given function represents the position of a particle travelling along a horizontal line.

(a) Find the velocity and accerelation functions.

(b) Determine the time intervals when the object is slowing down or speeding up.

(a) $s(t) = 2t^3 - 15t^2 + 36t - 10$

Velocity $= v(t) = s'(t) = \frac{d}{dt}(2t^3 - 15t^2 + 36t - 10)$

$= 6t^2 - 30t + 36.$

accerelation $= a(t) = v'(t) = \frac{d}{dt}(6t^2 - 30t + 36)$

$= 12t - 30.$

(b) $v(t) = 6t^2 - 30t + 36$

$v(t) = 0$ when $6t^2 - 30t + 36 = 0$

$t^2 - 5t + 6 = 0$

$(t-2)(t-3) = 0$

when $t = 2, t = 3$.

$t=0$ | $t=2$ | $t=3$

when $t>3$, $v(t)>0$, $a>0$.

when $t=1$: $v(1) > 0$, $a(1) < 0$

When $t = 2.5$: $v(2.5) < 0$, $a(2.5) < 0$

• When accerelation and velocity are in opposite direction, the object slows down.

• If $C(x)$ is the cost of producing x items ⇒ the marginal cost $MC(x)$ is $MC(x) = C'(x)$.

• If $R(x)$ is the revenue obtained from selling x items ⇒ the marginal ~~cost~~ revenue $MR(x)$ is $MR(x) = R'(x)$.

• If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items ⇒ the marginal profit $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$

16)

The price p (in dollars) and demand x for a certain ~~ditag~~ digital clock radio is given by the price-demand function $p(x) = 10 - 0.001x$.

(a) Find the revenue function $R(x)$.

(b) Find the marginal revenue function.

(c) Find the marginal revenue at $x = 2000$ & 5000.

(a) Revenue = Dollars obtained by selling x amount of clocks.

$\therefore R(x) = x\,p(x) = x(10 - 0.001x)$

(b) $MR(x) = \frac{d}{dx}[x(10 - 0.001x)]$

$= (10 - 0.001x) + x(-0.001)$

$= 10 - 0.002x.$

(c) $MR(2000)$

$MR(5000)$.

WK5

① $\frac{d}{dx}(\sin x) = \cos x$

② $\frac{d}{dx}(\cos x) = -\sin(x)$

③ $\frac{d}{dx}(\tan x) = \sec^2(x).$

$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{d}{dx}(\sin x)(\cos(x))^{-1}$

$= \cos(x)\cos(x)^{-1} + \sin(x)(-1)(\cos x)^{-2}(\sin x)$

$= 1 + (-1)\frac{\sin^2 x}{\cos^2 x}$

$= 1 - \frac{\sin^2 x}{\cos^2 x}$

$= 1 - \tan^2 x$

$= \sec^2 x.$

④ $\frac{d}{dx}\cot(x) = -\csc^2 x$

⑤ $\frac{d}{dx}(\sec x) = \sec x \tan x.$

⑥ $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

Qn 176

$y = 3\csc x + \frac{5}{x}$. Find $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{d}{dx}\left(3\csc x + \frac{5}{x}\right)$

$= \frac{d}{dx}(3\csc x) + \frac{d}{dx}\left(\frac{5}{x}\right)$

$= 3\frac{d}{dx}\csc x + \frac{d}{dx}5x^{-1}$

$= 3(-\csc x \cot x) + 5(-1)(x^{-2})$

$= -3\csc x \cot x - \frac{5}{x^2}$

Qn 183

$y = \frac{1-\cot x}{1+\cot x}$. Find $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-\cot(x)}{1+\cot(x)}\right)$

$= \frac{d}{dx}(1-\cot(x))(1+\cot(x))^{-1}$

$= (\csc^2 x)(1+\cot(x))^{-1} +$

$(1-\cot x)(-1)(1+\cot(x))^{-2}(-\csc^2 x)$

$= \frac{\csc^2 x}{1+\cot(x)} + \frac{\csc^2(x)(1-\cot(x))}{(1+\cot(x))^2}.$

Qn 186.

WK5

- Find the eqn of the tangent line at the given point.

$f(x) = \csc x \;, x = \frac{\pi}{2}.$

Slope of tangent line at $x=a$ = Derivative of function ($f'(a)$).

So, $f'(x) = \frac{df}{dx} = \frac{d}{dx} \csc(x)$

$= -(\csc(x))\cot(x)$

So, at $x = \pi/2$, $f(\pi/2) = \operatorname{cosec}(\pi/2) = 1$

$f'(\pi/2) = -\csc(\pi/2)\cot(\pi/2)$

$= -(1)\cdot 0$

$= 0.$

$\therefore\; y - 1 = 0\cdot(x - \pi/2)$

$\Rightarrow y = 1$

98) Find all the x values on the graph of $f(a) = x - 2\cos(x)$ for $0 < x < 2\pi$, where the tangent line has slope 2.

Slope of tangent line = Gradient of function ($f'(x)$)

$f'(x) = \frac{df}{dx} = \frac{d}{dx}(x - 2\cos(x)) = \frac{d}{dx}x - 2\frac{d}{dx}\cos(x)$

$= 1 + 2\sin x.$

So, we want to find all values of x s.t $1 + 2\sin x = 2.$

So, $1 + 2\sin x = 2 \iff 2\sin x = 1 \iff \sin(x) = 1/2.$

So, what are all the x values such that $\sin x = 1/2.$

$x = \sin^{-1}(1/2) = \pi/6$

~~But~~ So, $x = \pi/6 + 2k\pi,\; k \in \mathbb{Z}.$

Chain Rule

WK5

- When to use it? → When we have a function that is a composition of 2 (or more) functions.
- The Chain rule states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

E.g: $f(x) = \sin(x^3)$.

$\frac{dy}{dx} \leftarrow$ rate of Δ of $\sin(x^3)$ as x changes. (i.e how $\sin(x^3)$ changes relative to change in x.)

↳ So, we think of this as a chain reaction.

$$x\,\Delta \to \underbrace{x^3\,\Delta}_{\text{this tells us that the derivative of } x^3 \text{ is involved.}} \to \underbrace{\sin(x^3)\,\Delta}.$$

→ this suggest that the derivative of $\sin(u)$ wrt u is involved, where $u = x^3$ (as if $x^3\Delta \Rightarrow \sin(x^3)\Delta$).

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{d}{du}(\sin(u))\,\frac{d}{dx}u$$

$$= \frac{d}{du}\sin(u)\,\frac{d}{dx}x^3$$

$$= \cos(u)\,x^3 \quad \text{as } u = x^3$$

$$= \cos(x^3)\,x^2.$$

Steps: To differentiate $h(x) = f(g(x))$

1. Begin identifying $f(x)$ & $g(x)$
2. Find $f'(x)$ & evaluate it at $g(x)$, to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x)) \cdot g'(x)$

Another way to write Chain Rule using Leibniz's Notation:

+ $y = f(u)$ & $u = g(x)$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}.$$

Qn 214: Given $y = f(u)$, $u = g(x)$ find $\frac{dy}{dx}$ using $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{d}{du}(3u-6)\,\frac{d}{dx}(2x^2)$$

$$= 3\cdot(4x)$$

$$= 12x.$$

Qn 220: For each of the following exercises, decompose each function in the form $y = f(u)$ and $u = g(x)$ and find $\frac{dy}{dx}$ as a function of x.

$$y = (\underbrace{3x-2}_{u = 3x-2})^6$$

$$\therefore y = u^6,\ u = 3x-2.$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}(u^6)\cdot\frac{d}{dx}(3x$$

$$= 6u^5(3) = 18u^5 = 18(3x-2)^5.$$

Implicit Differentiation

WKS

- Implicit Differentiation allows us to find slopes of tangents to curves that are clearly not functions. (they fail the vertical line test.)

Steps

① Take derivatives of both sides of the equation. Keep in mind that y is a function of x. (So, whenever we differentiate y, we write $\frac{dy}{dx}$).

② Rewrite ~~all~~ the equation so that all the terms containing $\frac{dy}{dx}$ are on the left and all the terms that do not contain $\frac{dy}{dx}$ are on the right.

③ Factor out $\frac{dy}{dx}$.

④ Solve for $\frac{dy}{dx}$.

in 300

$x^2 - y^2 = 4$

Using implicit differentiation

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(4)$$

$$2x - 2y\frac{dy}{dx} = \cancel{4}\ 0$$

$$\frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}.$$

Qn 311

$x^2y^2 + 5xy = 14 \qquad (2, 1).$

$$\frac{d}{dx}(x^2y^2 + 5xy) = \frac{d}{dx}(14)$$

$$\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(5xy) = 0$$

$$2xy^2 + x^2 2\frac{dy}{dx} + 5y + 5x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x^2 + 5x) = -5y - 2xy^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(-5 - 2xy)}{x(5 + 2x)}. \#$$

So, gradient at $x = -1 = y$,

$$\frac{dy}{dx} = \frac{(-1)(-5-2)}{-1(5+2(-1))}$$

$$= \frac{7}{-3} \#$$

$$y - (-1) = -\frac{7}{3}(x - (-1)).$$

- For quantities that are changing wrt time, the rates at which these quantities change are given by derivatives.
- If 2 related quantities are changing over time, the rates at which the quantities change are related.

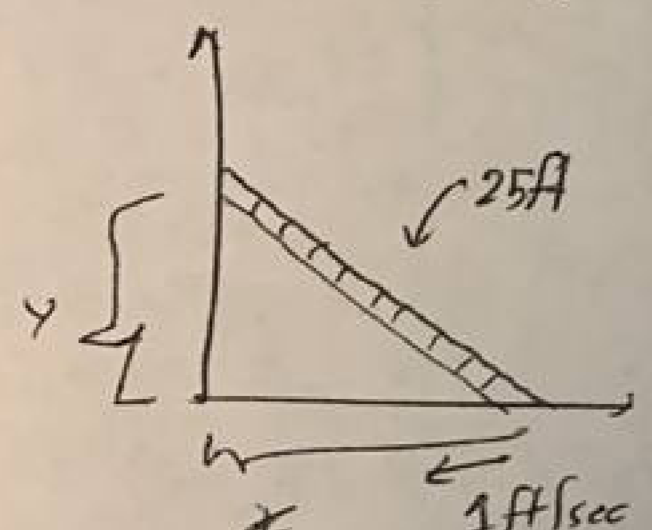
Wk 6

Problem-Solving Strategies

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an eqn relating the variables introduced in step 2
4. Using Chain rule, differentiate both sides of the equation in step 3 with respect to the independent variable. This new eqn will relate the derivatives.
5. Substitute all known values into the eqn from step 4 and then solve for the unknown rate of change.

Qn 6.

A 25ft ladder is leaning against the wall. If we push the ladder towards the wall at a rate of 1ft/sec, and the bottom of the ladder is initially 20ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?



Let x be horizontal distance from the wall.
Let y be the vertical distance from the wall.

Then, it is given that $\frac{dx}{dt} = 1$ ft/sec.

$$x^2+y^2 = 25^2 = 625.$$

$$y = \sqrt{625-x^2}.$$

We need to find $\frac{dy}{dt}$.

$$\frac{dy}{dx} = \frac{1}{2}(625-x^2)^{-1/2}(-2x)$$

$$= -x(625-x^2)^{-1/2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= -x(625-x^2)^{1/2}.$$

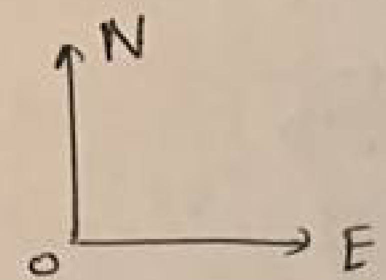
After 5 sec of pushing, $x = 15$.

$$\therefore \frac{dy}{dt} = -15(625-15^2)^{-1/2}$$

$$= -15\cancel{(20)}\frac{1}{20}$$

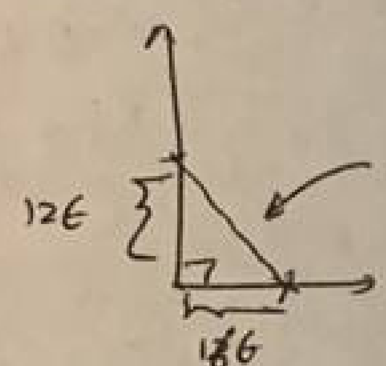
$$= -\frac{15}{20}$$

(3) You and a friend are riding your bikes to a restaurant that you think is east, Your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16mph east and your friend riding 12mph north.
After you travelled 4 miles, at what rate is the 'distance between you changing? WK6



At time t hours, Your dist from origin O is $16t$.
Friend's dist from origin O is $12t$.

Distance between you and your friend,



$$\sqrt{(16t)^2+(12t)^2} = \sqrt{400t^2}$$
$$= 20t.$$
$$\frac{dD}{dt} = 20t.$$

~~After 4hrs~~ $\cancel{\frac{dD}{dt} = 20 \times 4 = 80}$ At 4 miles, time passed is 15 mins $= \frac{1}{4}$ hrs.

$$\therefore \left.\frac{dD}{dt}\right|_{t=1/4} = 5 \text{ miles}.$$

(20) The radius of a sphere increases at a rate of 1m/sec. Find the rate at which the volume increases when the radius is 20m.

$$\frac{dr}{dt} = 1\text{m/sec}.$$

$$V = 4\pi r^3$$
$$\frac{dV}{dr} = 12\pi r^2$$
$$\frac{dV}{dr} = \frac{dV}{dt} \cancel{\times} \div \frac{dr}{dt}$$

when $r = 20$, $\frac{dV}{dr} = 4800\cancel{\pi}$

$$\frac{dV}{dt} = 4800$$

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Derivative of natural Exponential function

Let $E(x)=e^x$ be the natural function. Then,

$$E'(x)=e^x.$$

In general,

$$\frac{d}{dx}e^{g(x)}=g'(x)e^{g(x)}.$$

Derivative of Logarithmic function:

If $x>0 \wedge y=\ln(x) \Rightarrow \frac{dy}{dx}=1/x.$

Given $h(x)=\ln(g(x)) \Rightarrow h'(x)=g'(x)\frac{1}{g(x)}.$

WK6

Problem-Solving Strategy

→ Using Logarithmic Differentiation.

1. To differentiate $y=h(x)$ using LD, take natural logarithm of both sides of the equation to obtain $\ln(y)=\ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation.
4. Make $\frac{dy}{dx}$ the subject.
5. Replace y by $h(x)$.

Qn 333 Sec 3.9

$f(x)=e^{x^3\ln(x)}$

$$f'(x)=\frac{d}{dx}e^{x^3\ln(x)}$$

$$=\left[\frac{d}{dx}x^3\ln(x)\right]e^{x^3\ln(x)}$$

$$=\left[3x^2\ln(x)+x^3\left(\frac{1}{x}\right)\right]e^{x^3\ln(x)}$$

$$=\left(3x^2\ln(x)+x^2\right)e^{x^3\ln(x)}$$

Qn 338 Sec 3.9

$f(x)=3^{\sin(3x)}$

Let $y=3^{\sin(3x)}$

$\ln(y)=\sin(3x)\ln(3)$

Differentiating wrt y,

$$\frac{1}{y}\frac{dy}{dx}=3\cos(3x)\ln(3)$$

$$\frac{dy}{dx}=3\ln(3)y\cos(3x).$$

Since $y=3^{\sin(3x)}$

$$\frac{dy}{dx}=3\ln(3)3^{\sin(3x)}\cdot\cos(3x)$$

Qn 346.

$y=x^{\sqrt{x}}.$

$\ln(y)=\ln x^{\sqrt{x}}$

$\ln(y)=\sqrt{x}\ln(x)$

Differentiate wrt x,

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2\sqrt{x}}\ln(x)+\sqrt{x}\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx}=y\left(\frac{1}{2\sqrt{x}}\ln(x)+\frac{1}{\sqrt{x}}\right)$$

$$\Rightarrow \frac{dy}{dx}=x^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln(x)+\frac{1}{\sqrt{x}}\right).$$

Wk 6

Derivative of Inverse functions

Thm: let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$.

$\forall x$ satisfying $f'(f^{-1}(x)) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = (f^{-1})'(x)$$

$$= \frac{1}{f'(f^{-1}(x))}$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$,

$$g'(x) = \frac{1}{f'(g(x))}.$$

Derivatives of trigo inverse functions:

1. $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Qn 280

Method ①: $y = \cos^{-1}(\sqrt{x})$

$\cancel{\frac{dy}{dx}}$ Let $u = \sqrt{x}$.

$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\therefore \frac{dy}{dx} = \frac{d}{du}\left(\cos^{-1}(u)\right)\frac{d}{dx}u$

$= \frac{d}{du}\cos^{-1}(u)\,\frac{d}{dx}\sqrt{x}$

$= \frac{-1}{\sqrt{1-u^2}} \cdot \frac{1}{2}x^{-1/2}$

$= -\frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x}\sqrt{1-x}}$

Method ②: $y = \cos^{-1}(\sqrt{x})$

$\cancel{= [\cos(\sqrt{x})]^{-1}}$

$\cancel{\frac{dy}{dx} = -1}$

$\cos(y) = \sqrt{x}$.

$-\sin(y)\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$

$\therefore -\sqrt{1-x}\,\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$

Sec 3.7 Qn 280

Find $\frac{dy}{dx}$ using the given function.

$y = \cos^{-1}(\sqrt{x}) = f(x)$

$y^{-1} = \cos^2(x) = g(x). \Rightarrow \frac{d(y^{-1})}{dx} = -2\cos(x)\sin(x).$

$\therefore \cancel{\frac{dy}{dx}=}$ $\frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1}(\sqrt{x})\right]$

$= \frac{1}{-2\cos(\cos^{-1}(\sqrt{x}))\sin(\cos^{-1}(\sqrt{x}))} = \frac{1}{-2\sqrt{x}\sin(\cos^{-1}\sqrt{x})}$

Qn 281

$f(x) = y = \sec^{-1}(1/x)$

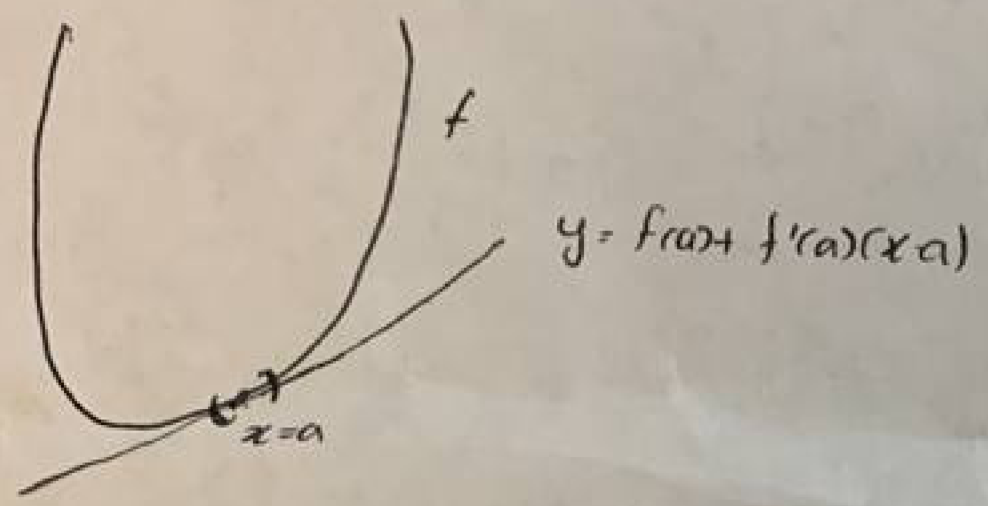
$f^{-1}(x) = \sec(1/y)$

Wk 7

- Derivatives can be used to approximately functions locally by linear function
- Recall that if f is differentiable at a point $x=a$, the tangent line to the graph of f at a is given by the equation:

$$y = f(a) + f'(a)(x-a)$$

- Note that for a given function f at $x=a$, near $x=a$, the graph of the tangent line is close to the graph of f.



- As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near (a).
- In general, for a differentiable function f, the equation of the tangent line to f at $x=a$ can be used to approximate $f(x)$ for x near a. Thus, we can write

$$f(x) \approx f(a) + f'(a)(x-a) \text{ for } x \text{ near } a.$$

- We call the linear function $L(x) = f(a) + f'(a)(x-a)$ the linear approximation, or tangent line approximation, of $x=a$. The function L is known as the linearisation of f at $x=a$.

Qn 51 Sec 4.2

$f(x) = \frac{1}{x}$. Find the linear approximation $L(x)$ to $y=f(x)$ near $x=a$ for the function, $a=2$.

- So, we need to find the equation of the tangent line at $x=a$. =

$$f'(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx} x^{-1} = -x^{-2} = -\frac{1}{x^2}.$$

So, $f'(2) = -\frac{1}{4}$.

$(a, f(a)) \mapsto (2, f(2)) = (2, \frac{1}{2})$.

Eqn: $y - \frac{1}{2} = -\frac{1}{4}(x-2)$.

Qn 53

$f(x) = \sin(x)$, $a = \pi/2$

$f'(x) = \cos(x)$

$f'(a) = f'(\pi/2) = \cos(\pi/2) = 0$

$(a, f(a)) = (\pi/2, \sin(\pi/2)) = (\pi/2, 1)$

$y - 1 = 0(x - \pi/2)$

$\therefore y = 1.$

Qn 58: Compute ~~sec(0.03)~~ $\cos(0.03)$ within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x-a)$.

Let $f(x) = \cos(x)$

$\therefore f'(x) = \sin(x)$

Let $a = 0.02$.

$f'(a) = f'(0.02) = 0.0\dot{1}9999 \approx 0.02$.

$(a, f(a)) \mapsto (0.02, \cos(0.02)) = 0.9998 \approx 1$.

$y - 1 = 0.02(x - 0.02)$

So, when $x = 0.03 \Rightarrow y - 1 = 0.02(0.01)$

$\Rightarrow y = 1.0002$

- Linear ~~functions~~ approximations can be used to estimate function value. They can also be used to estimate a small change of the amt a function you changes as a result of in the input.

- Concept is formally known as differentials.
  - Looking at the derivative = $\frac{dy}{dx} = f'(x)$.

WK7

$$\Rightarrow dy = f'(x)\, dx.$$

So, dy is a function of both x and dx.
dy, dx are called differentials.

- We connect differentials to linear approximation. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values.
Suppose that the function f is differentiable at point $x=a$. Suppose the input x changes by a small amount. We are interested in how much the output y changes.
If x changes from a to $a+dx \Rightarrow$ the change in x is dx, i.e. $\Delta x = dx$, and change in y,

Δy is given by $\Delta y = f(a+dx) - f(a)$.

- Instead of calculating the exact change in y, it is often easier to approximate the change in y by using a linear approximation. For x near a, $f(x)$ is approximated by

$$L(x) = f(a) + f'(a)(x-a).$$

$\therefore$ If dx is small, $f(a+dx) \cong L(a+dx) = f(a) + f'(a)(a+dx-a)$

$$\text{i.e. } \underbrace{f(a+dx) - f(a)}_{\text{actual } \Delta \text{ in } y} \cong \cancel{f'(a)\,dx}.\ L(x) - f(a) = \underbrace{f'(a)\, dx}_{\text{approximated change in } y}.$$

In other words, the actual change in the function f, if x increases from a to $a+dx$, is approximately the difference between $L(a+dx)$ and $f(a)$. By definition of $L(x)$, this difference is $f'(a)\,dx$.
In summary, $\Delta y = f(a+dx) - f(a) \cong L(a+dx) - f(a) = f'(a)\, dx$.

Qn 72 Find differential and evaluate for the given x and dx.

$y = 3x^2 - x + 6$, $x = 2$, $dx = 0.1$

$$dy = f'(x)\,dx$$
$$= \left(\frac{d}{dx} 3x^2 - x + 6\right) dx$$
$$= (6x-1)\, dx.$$

$\rightarrow$ So, when $x=2$, $dx = 0.1$,

$$dy = (12-1)\,0.1$$
$$= 11 \times \frac{1}{10}$$
$$= 1.1$$

Calculating the amount of Error

(Wk7)

- Any type of measurement is prone to a certain amt of error.
- For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area.
- Differentials can be used to estimate the error.

Defi: $dy = f'(x)\,dx$

→ Consider the function f with an input that is a measured quantity. Suppose the exact value is a, but the measured value is $a+dx$.

So, the measurement error is dx (or Δx).

As a result, an error occurs in the calculated quantity $f(x)$.

$$\Delta y = f(a+dx) - f(a).$$

But because, we do not know the exact value of a measured quantity (all measurements are prone to some degree of error), we can't calculate the propagated error exactly.

However, given an estimate of the accuracy of the measurement, we can use differentials to approximate the propagated error. Δy.

Specifically if f is differentiable at a,

$$\Delta y \approx dy = f'(a)\,dx.$$

Since we do not know the exact value a, we can use the measured value $a+dx$,

$$\Delta y \approx dy \approx f'(a+dx)\,dx.$$

Qn 80 Find the change in ~~Volume~~ Area ~~dv~~ dA if radius of a sphere changes from r to dr.

~~ra~~

Area of sphere $= 4\pi r^2$

$\therefore \dfrac{dA}{dr} = 8\pi r$

$\therefore dA = 8\pi r\,dr.$

$\Delta y = 8\pi(r+dr)\,dr$
$= 8\pi(rdr + dr^2)$

↓

Intuitive way

① $r \mapsto r - dr.$

Area $= 4\pi(r-dr)^2$
$= 4\pi(r^2 - 2rdr + dr^2)$

② $r \mapsto r+dr$

Area $= 4\pi(r+dr)^2$
$= 4\pi(r^2 + 2rdr + dr^2)$
$= 4\pi r^2 + 8\pi rdr + 4\pi dr^2$

Change in Area

$4\pi r^2 - 4\pi r^2 + 4\pi 2dr - 4\pi dr^2 \leq A \leq 4\pi r^2 - 4\pi r^2 - 4\pi rdr - 4\pi dr^2$

$r\,4\pi dr - 4\pi dr^2 \leq A \leq -4\pi rdr - 4\pi dr^2$

$\Rightarrow 4\pi(rdr - dr^2) \leq A \leq 4\pi(-rdr - dr^2)$

Wk7

Maxima / Minima

- Finding the maximum and minimum of the function. (Used to solve optimization problem)
- Definition: Let f be a function defined over an interval I & let $c \in I$. We say f has an absolute maximum on I at c if $f(c) \geq f(x), \forall x \in I$. We say that f has an absolute minimum on I at c if $f(c) \leq f(x), \forall x \in I$.

  If f has an absolute maximum on I at c or an absolute ~~maximum~~ minimum on I at c, we say f has an absolute extremum on I at c.
- Extreme Value theorem: If f has a ctn function over the closed, bounded interval $[a,b]$, then there is a point in $[a,b]$ at which f has an absolute maximum over $[a,b]$ & $\exists$ a point at which f has an absolute minimum over $[a,b]$.
- Definition of Critical Point: Let c be an interior point in the domain of f. We say that c is a critical point of f if $f'(c) = 0$ or $f'(c)$ is undefined.
- Location of Absolute Extrema: Let f be a ctn function over a closed, bounded interval I. The absolute maximum of f over I and the absolute ~~value~~ minimum of f over I must occur at endpoints of I or at critical points of f in I.

Problem-Solving strategy

1. Evaluate f at the endpoints $x=a$ and $x=b$.
2. Find all critical points of f that lie over the interval (a,b) and evaluate f at those critical points.
3. Compare all the values found in (1) and (2). From location of absolute extrema, the absolute extrema must occur at the endpoints or critical points.

Qn 109. Find the critical point in the domain of the function $4\sqrt{x} - x^2 = y$.

$$\frac{dy}{dx} = \frac{d}{dx}(4\sqrt{x} - x^2)$$
$$= 4\left(\tfrac{1}{2}\right)x^{-1/2} - 2x$$
$$= 2x^{-1/2} - 2x$$
$$= \frac{2}{\sqrt{x}} - 2x.$$

Critical pts: $\frac{dy}{dx} = 0$. ie $\frac{2}{\sqrt{x}} - 2x = 0$

$$\Leftrightarrow \frac{2}{\sqrt{x}} = 2x$$
$$\Leftrightarrow 2 = 2x\sqrt{x}$$
$$\Leftrightarrow 1 = x^{3/2}$$
$$\Leftrightarrow x = 1$$

Qn 120 Find the local and/or absolute maxima for the functions.

$$y = (x - x^2)^2 \text{ over } [-1, 1].$$

(1) $y(-1) = (-1 - (-1)^2)^2 = (-1-1)^2 = 4.$

$y(1) = (1 - (1)^2) = 0$

(2) $y = (x - x^2)^2 \quad \frac{dy}{dx} = 2(x - x^2)\left[\frac{d}{dx}(x - x^2)\right]$
$$= 2(x - x^2)(1 - 2x)$$

$\frac{dy}{dx} = 0 \Leftrightarrow (x - x^2)(1 - 2x) = 0$

$\Leftrightarrow \underbrace{x - x^2 = 0}_{x(1-x)=0}$ or $\underbrace{1 - 2x = 0}_{x = 1/2}$

$\therefore x = 0, \tfrac{1}{2}$ or 1.

$f(x) = 0 \quad \tfrac{1}{16} \quad 0$

$\therefore$ Max is at $x = -1$.

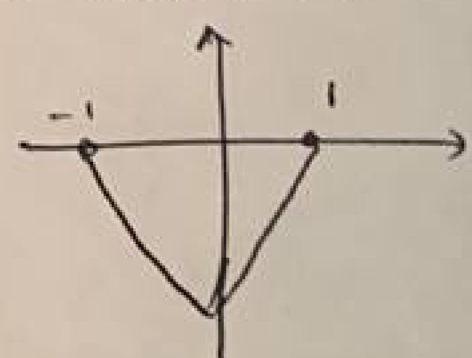
① Rolle's Theorem
② Mean Value theorem.

WK8

Rolle's theorem: Let f be a continuous function over the closed interval $[a,b]$ and differentiable over the open interval (a,b) such that $f(a)=f(b)$. Then, there exist at least one $c \in (a,b)$ s.t $f'(c)=0$.

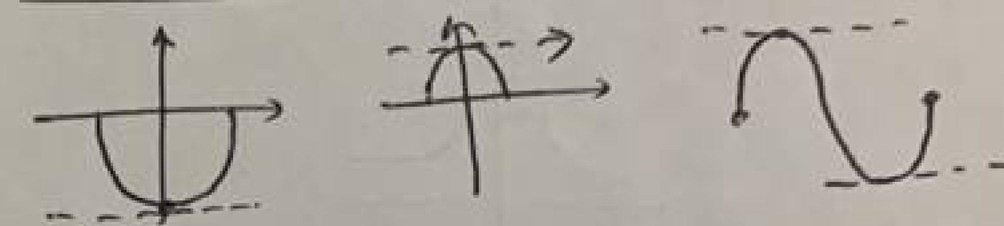
→ What happens when f is not differentiable? Then, there will be functions where $f(a)=f(b)$ but no $c \in (a,b)$ s.t $f'(c)=0$.

Consider: $f(x)=|x|-1$



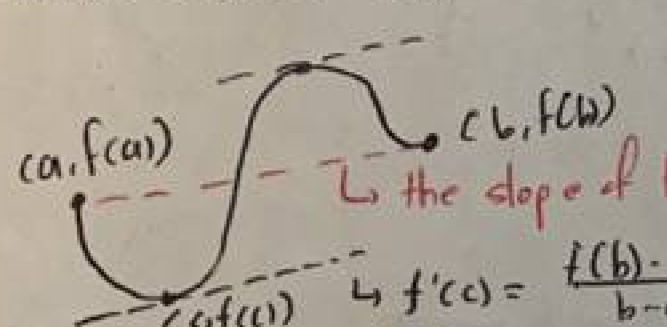
This is not differentiable over $[-1,1]$, specifically at $x=0$. It is cts over $[-1,1]$ and $f(-1)=0=f(1)$.
But $\forall c \in (-1,1)$, $f'(c) \neq 0$.

Examples where Rolle's theorem work



Mean value theorem

• This is a generalisation of Rolle's theorem: In rolle's theorem, we need the endpoints of the interval to be the same under f, ie given $[a,b]$, we need $f(a)=f(b)$.
The MVT generalises RT by considering functions that do not have equal values at endpoints.
ie MVT is the "slanted" version of RT.



↳ the slope of this line $= \frac{y_2-y_1}{x_2-x_1} = \frac{f(b)-f(a)}{b-a}$

↳ $f'(c) = \frac{f(b)-f(a)}{b-a}$

MVT: Let f be cts over the closed interval $[a,b]$ and differentiable over the open interval (a,b). Then, there exist at least one point $c \in (a,b)$ s.t

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Qn 162

Use the MVT and find all points $0<c<2$ such that $f(2)-f(0)=f'(c)(2-0)$, where $f(x)=\sin(\pi x)$.

Soln.

It is given that $f(2)-f(0)=f'(c)(2-0)$.

By rearranging the eqn above, $\frac{f(2)-f(0)}{2-0}=f'(c)$

As $f(x)=\sin(\pi x)$, $f(0)=\sin(\pi \cdot 0)=\sin(0)=0$ and
$f(2)=\sin(\pi \cdot 2)=\sin(2\pi)=0$.

So, actually we can use RT here too!!

So, we want to find $c \in (0,2)$ s.t $f'(c)=\frac{f(2)-f(0)}{2-0}=\frac{0-0}{2-0}=\frac{0}{2}=0$.

Since $f(x)=\sin(\pi x)$, $f'(x)=\pi\cos(\pi x)$ (By chainrule. Let $u=\pi x$.)

So, we want to solve for x such that $\pi\cos(\pi x)=0$

ie $\cos(\pi x)=0$, where $x \in (0,2)$.

$\pi x=\cos^{-1}(0)=\pi/2 \quad \therefore x=1/2$.

Qn 168

Wk3

For the function ~~$t62$~~ $f(x) = \frac{1}{x^2}$, show that there is no c s.t $f(1) - f(-1) = f'(c)(2)$.
Explain why the MVT doesn't apply here. in the interval $[-1, 1]$.

Soln. For MVT to apply the impt conditions are:

① f be ctn over $[a, b]$ (in this case $[-1, 1]$)

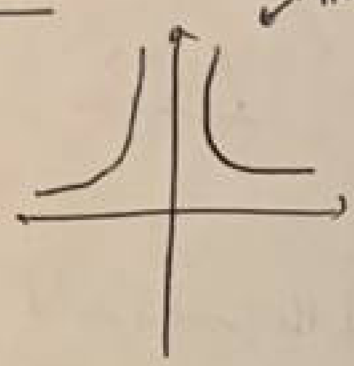
② f be differentiable ~~at~~ over (a, b) (In this case $(-1, 1)$).

Note that $f(x) = \frac{1}{x^2}$ is not defined at $x = 0$ as we can't divide 1 by 0. i.e $\frac{1}{0}$ doesn't exist.

Thus, $f(x)$ is not ctn ~~at~~ over $[-1, 1]$.

Graph of $f(x)$

infinite discontinuity.



~~Qn 141~~

~~2 cars drive from one spotlight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.~~

Qn 190

At 10.17am, you pass a police car of 55mph that is stopped on the ~~[illegible]~~ freeway. You pass a 2nd police car of 55mph at 10.53am, which is located 39mile from the first car. If the ~~[illegible]~~ speed limit is 60mph, can the police cite you for speeding?

Soln.

Time taken to reach the 2nd car to 1st car: 38mins. $= \frac{38}{60}$ hrs.

Distance travelled = 39 miles.

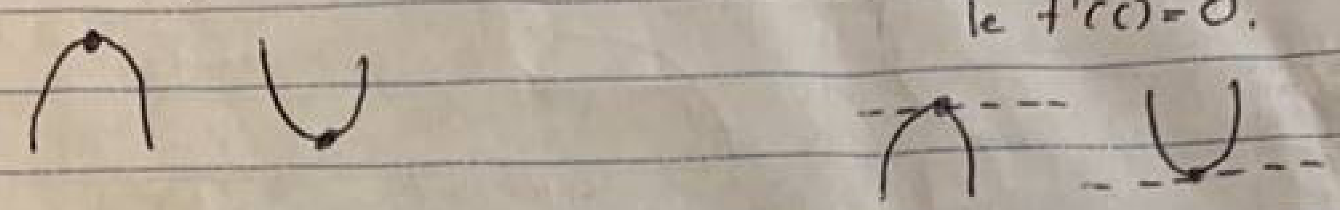
Average ~~time taken~~ speed $= \frac{\text{total Distance}}{\text{total time taken}} = \frac{39}{\left(\frac{38}{60}\right)} = 39 \times \frac{60}{38} = 61.6$ mph.

So, for average speed to be 61.6mph, speed limit has to be exited.

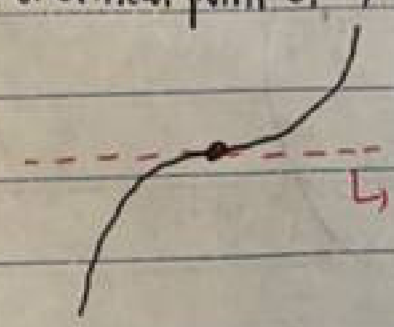
WK8

• We know that if f has a local maximum/minimum at c $\Rightarrow$ c must be a critical point of f.

ie $f'(c)=0$.



However, the converse is not true. At a critical point c, f may not have a local extremum. Consider

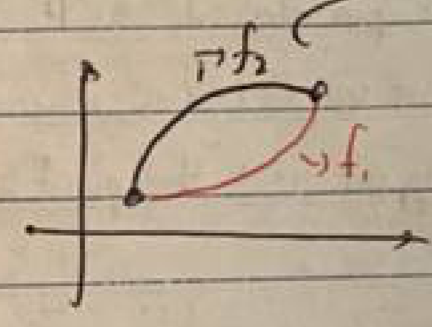


↳ $f'(c)=0$, but $f(c)$ is neither the max nor the min.

Goal: Determine whether a critical point of a function actually corresponds to a local extreme value.

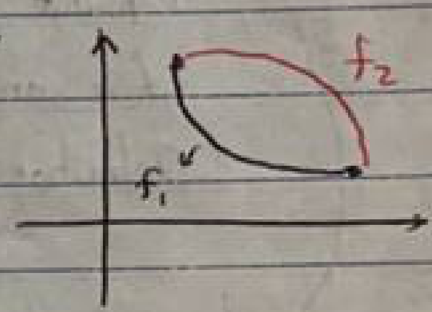
① The first Derivative Test. ← tells us when a function is increasing or decreasing

$x_1>x_2 \Rightarrow f(x_1)>f(x_2)$ $\qquad$ $x_1>x_2 \Rightarrow f(x_1)<f(x_2)$



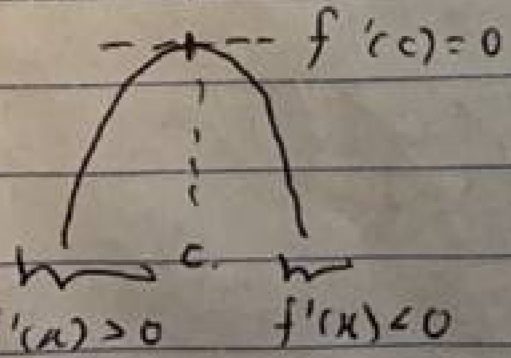
we have 2 types of increasing functions

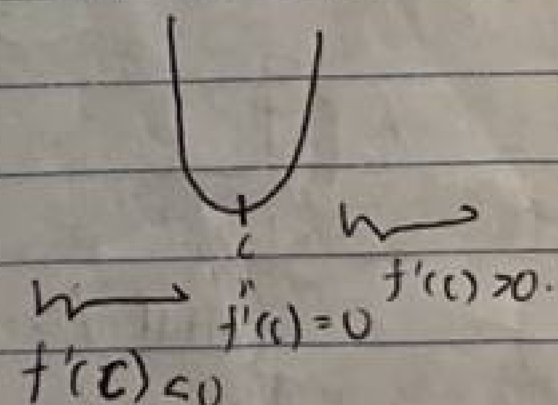
we have 2 types of decreasing function.



$f'(x)>0$ $\qquad$ $f'(x)<0$.

When does a function have a local max? $\qquad$ When does f have a local min?





Strategy to find local min/max using FDT:

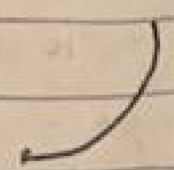
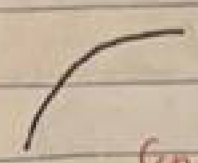
① Find all the critical pts.

② Analyze f' in each subinterval.

③ Use FDT.

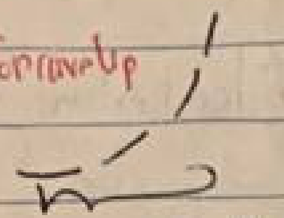
W1C8

Now, we want to tell apart the ↗ increasing and ↘ decreasing function?

 vs 

Concave down

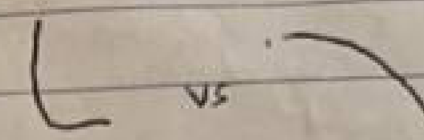
Concave Up

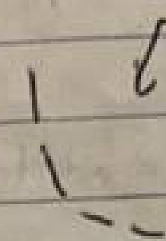
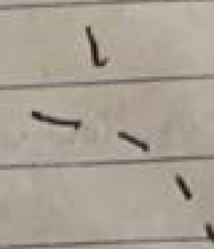


$\left.\right\} f''>0$

→ We see that $f'(x)$ increases as x increases

we see that $f'(x) \downarrow$ as $x \uparrow$. $\left.\right\}$ as $f''<0$

 vs Concave down

$f' \uparrow$ as $x \uparrow$.
$f''>0$
Concave Up

$f' \downarrow$ as $x \uparrow$.
$f''<0$.

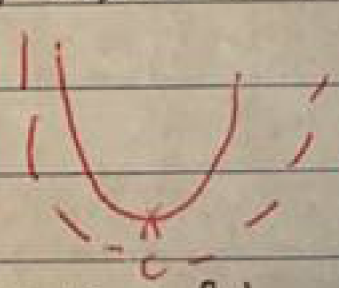
2nd Derivative test: Tells us the Δ of change of f'.

Definition: Let f be a function that is differentiable over an open interval I. If f' increasing over $I \Rightarrow$ f is concave up over I. If f' is decreasing over I, we say that f is concave down over I.

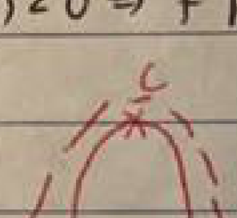
Definition: If f is continuous at a and f changes concavity at $a \Rightarrow$ the point $(a, f(a))$ is an inflection point of f.

Suppose that $f'(c)=0$, f'' is ctn over an interval containing c.

(i) If $f''(c)>0 \Rightarrow f$ has a local min at c.



(ii) If $f''(c)<0 \Rightarrow f$ has a local max at c.



(iii) If $f''(c)=0 \Rightarrow$ the test is inconclusive.

WK8

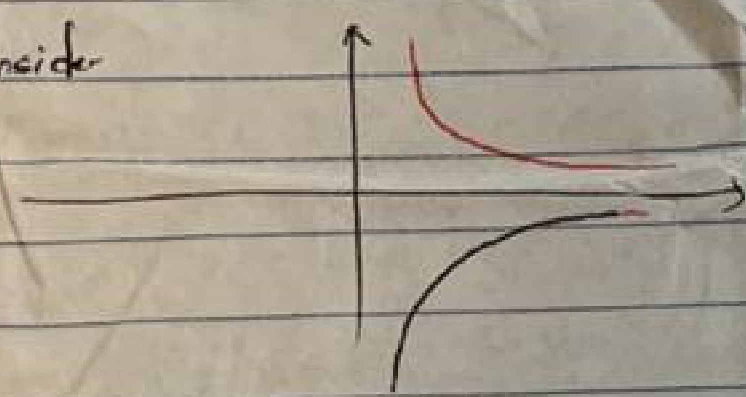
(194) Explain whether a concave-down function has to cross $y=0$ for some value of x.

Soln

For a function to concave downwards, $f''(x) < 0$. ie rate of Δ of derivative must be less than 0.

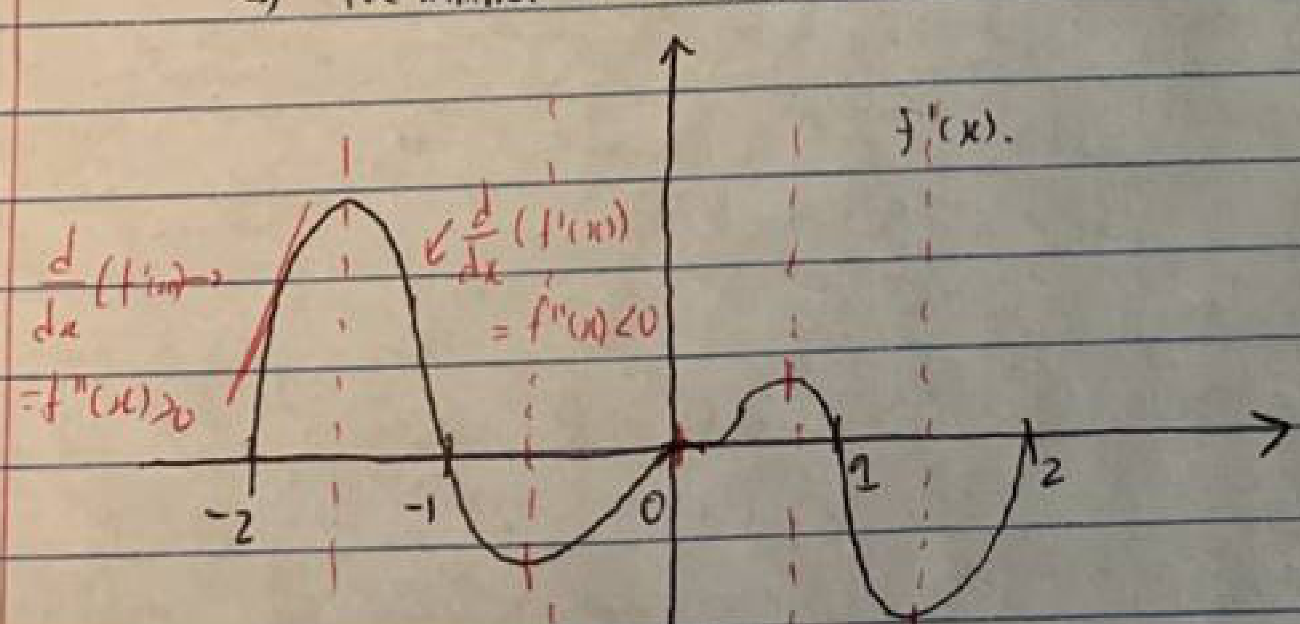
for an increasing function, for a decreasing function

The answer is nope. Consider



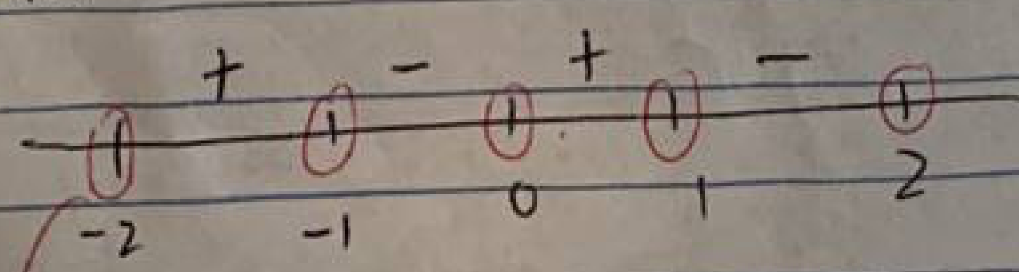
← $y=0$ acts as an asymptote.

(204) For the following exercises, analyse the graph of f', then list all ~~#~~ intervals where
(a) f is increasing and decreasing and
(b) the minima and maxima is located.
(c) List all the inflection points and
(d) Intervals f that are concave up/down



(c) Note that the graph is the derivative graph. $y = f'(x)$
$\therefore y' = f''(x)$

$(-2,-1) \Rightarrow f'(x) > 0 \Rightarrow f$ is increasing
$(-1,0) \Rightarrow f'(x) < 0 \Rightarrow f$ is decreasing
$(0,1) \Rightarrow f'(x) > 0$
$(1,2) \Rightarrow f'(x) < 0$



$f'(c) = 0$. $\therefore$ critical pts.

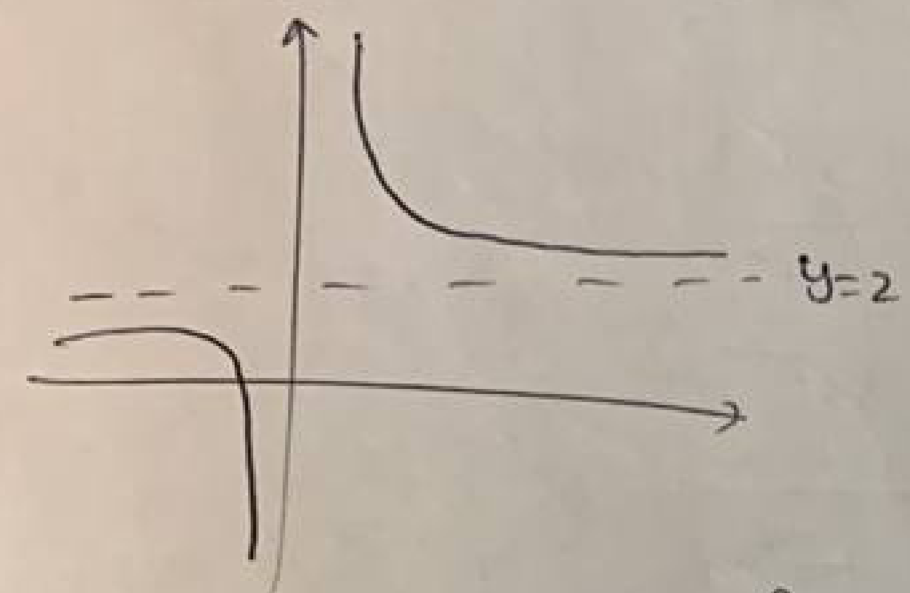
Out of these 5 pts, 3 of them are inflection pts.

• Recall that $\lim_{x \to a} f(x) = L$ means that $f(x)$ gets closer and closer to L as x gets closer and closer to a.

• So, when we have $\lim_{x \to \infty} f(x) = L$, this means that $f(x)$ gets closer and closer to L as x gets closer and closer to infinity (which is the same thing as ~~x~~ "as x gets larger and larger").

UKa

Consider the function: $y = 2 + \frac{1}{x}$



• Horizontal Asymptote: If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then we say that the line $y = L$ is a horizontal asymptote of f.

Computing Limits at Infinity

techniques: Limit Laws (Section 2.3); Squeeze theorem (Section 2.3)

Qn For the following exercises, evaluate the limit.

(261) $\lim_{x \to \infty} \frac{1}{3x+6} = 0.$

(263) $\lim_{x \to \infty} \frac{x^2 - 2x + 5}{x+2}$. (Notice that degree of numerator > degree of the denominator).

Roots of $x^2 - 2x + 5$

$-(-2) \pm \underbrace{\sqrt{4 - 4(1)(5)}}_{<0.}$

$\therefore$ We cannot factorise $x^2 - 2x + 5$.

$$
\begin{array}{r}
x - 4 \\
x+2 \overline{\big) x^2 - 2x + 5} \\
-(x^2 + 2x) \\
\hline
-4x + 5 \\
-(-4x - 8) \\
\hline
13 \\
\hline
\end{array}
$$

$\therefore \frac{x^2 - 2x + 5}{x+2} = (x-4) + \frac{13}{x+2}$.

$\therefore \lim_{x \to \infty} (x-4) + \frac{13}{x+2}$ } Also not very helpful.

$$\lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)(x^2 - 2x + 5)}{\left(\frac{1}{x}\right)(x+2)}$$

$$= \lim_{x \to \infty} \frac{x - 2 + 5/x}{1 + 2/x}.$$

$\therefore$ Denom $\to 1$

Num $\to \infty$.

$\therefore \lim_{x \to \infty} \frac{x^2 - 2x + 5}{x+2} = \infty.$

(268) $\lim_{x\to\infty} \frac{4x}{\sqrt{x^2-1}}$

$= \lim_{x\to\infty} \frac{(\frac{1}{x})4x}{(\frac{1}{x})\sqrt{x^2-1}}$

$= \lim_{x\to\infty} \frac{4}{\sqrt{(x^2-1)\frac{1}{x^2}}}$

$= \lim_{x\to\infty} \frac{4}{\sqrt{1-\frac{1}{x^2}}}$

$= \frac{\lim_{x\to\infty} 4}{\sqrt{\lim_{x\to\infty}(1-\frac{1}{x^2})}} = \frac{4}{1} = 4$#.

(269) $\lim_{x\to-\infty} \frac{4x}{\sqrt{x^2-1}}$

$= \frac{\lim_{x\to-\infty} 4}{\sqrt{\lim_{x\to-\infty}(1-\frac{1}{x^2})}}$

$= \frac{4}{\sqrt{1}} = 4$#.

(270) $\lim_{x\to\infty} \frac{2\sqrt{x}}{x-\sqrt{x}+1}$

$= \lim_{x\to\infty} \frac{(\frac{1}{\sqrt{x}})2\sqrt{x}}{(\frac{1}{\sqrt{x}})(x-\sqrt{x}+1)}$

$= \lim_{x\to\infty} \frac{2 \to 2.}{\underbrace{\sqrt{x}-1+\frac{1}{\sqrt{x}}}_{\to\infty}}$

$= 0.$

Find the horizontal and vertical asymptotes. for the following exercises.

(271). $f(x) = x - \frac{9}{x}$.

VKa

$x = 0$ ← vertical asymptote.

Oblique Asymptote $= y = x$

(273) $f(x) = \frac{x^3}{4-x^2}$

$\cancel{f(x)=}$ $4-x^2 \overline{)-x^3}$ $-x$

$\frac{-(-4x+x^3)}{4x}$.

$f(x) = -x + \frac{4x}{4-x^2}$.

Oblique asymptote: $y = -x$.

Vertical asymptote => $4-x^2=0$

$(x+2)(x-2)=0$.

(274) $f(x) = \frac{x^2+3}{x^2+1}$

$x^2+1 \overline{)x^2+3}$ 1

$\frac{-(x^2+1)}{2}$

$f(x) = 1 + \frac{2}{x^2+1}$

← Never zero.

$\therefore y = 1$

$x^2+1=0 \Leftrightarrow x^2 = \sqrt{-1}$ (Not possible).

Guideline for Drawing the Graph of a Function

WKA

General Strategy: Given a function f, use the following steps to sketch a graph of f:

1. Determine the domain of the function.
2. Locate the x and y intercepts.
3. Evaluate the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the end behaviour. If either of these limits is a finite number L, then $y=L$ is a horizontal asymptote. If either of those limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x)=\frac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as

$$f(x)=\frac{p(x)}{q(x)}=g(x)+\frac{r(x)}{q(x)},$$

where $\deg(r(x)) < \deg(q(x))$. The values of $f(x)$ approach the values of $g(x)$ as $x\to\pm\infty$. If $g(x)$ is a linear function, it is known as an oblique asymptote.

4. Determine whether f has any vertical asymptotes.
5. Calculate f'. Find all the critical pts and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
6. Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Qn: For the following exercises, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points and asymptotic behavior.

(294) $y=3x^2+2x+4$.

(1) Domain: $\mathbb{R}$ as $y(x)$ is a polynomial.

(2) When $x=0$, $y=4$. $(0,4)$ ← y-intercept.

When $y=0$, $0=3x^2+2x+4$

Discriminant $= b^2-4ac = 4-4(3)(+4)$
$= 4-48$
$= -44 < 0$.

∴ No real soln.

So, there is no x intercepts.

(3) $\lim_{x\to\infty} 3x^2+x+4=\infty$.

$\lim_{x\to-\infty} 3x^2+x+4=\infty$.

(4) No vertical asymptotes.

(5) $f'=6x+2$.

$f'(x)=0 \iff 6x+2=0$
$x=-\frac{1}{3}$.

Domain $\mathbb{R}$.
Split the domain into 2 parts
$(-\infty,-\frac{1}{3})(-\frac{1}{3},\infty)$

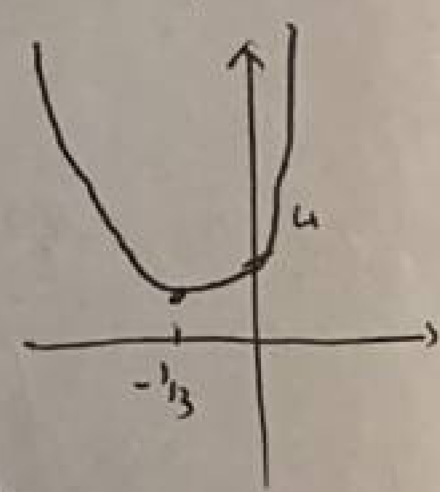
Pick 2 points -1, 0.

x	-1	$-\frac{1}{3}$	0
$f'(x)$	$f'(-1) = -6+2 = -4<0$.	0.	$f'(0)=6(0)+2 =2>0$
	<0		
	\	—	/

∴ $(-\frac{1}{3},0)$ is a local min

(6) $f''(x)=6>0$

∴ Graph concave upwards.



(297) $y = \dfrac{x^3+4x^2+3x}{3x+9}$.

(1) Domain: y is undefined ~~at~~ when denominator $= 0$.

ie $3x+9=0$.

$\Rightarrow 3x = -9$

$\Rightarrow x = -3$.

$\therefore$ Domain is : $\mathbb{R} / \{-3\}$.

(2) When $y=0$, $\dfrac{x^3+4x^2+3x}{3x+9} = 0$.

$x^3+4x^2+3x=0$.

$x(x^2+4x+3)=0$

$x(x+1)(x+3)=0$

$\therefore \underbrace{x=0,\ x=-1,\ x=-3}_{x\text{-intercepts.}}$

When $x=0$, $y = \frac{0}{9} = 0$.

$\therefore$ y-intercept $(0,0)$.

(3) $\lim\limits_{x\to\infty} \dfrac{x^3+4x^2+3x}{3x+9} = \lim\limits_{x\to\infty} \dfrac{x(x+1)\cancel{(x+3)}}{3\cancel{(x+3)}}$

$= \lim\limits_{x\to\infty} \dfrac{x(x+1)}{3}$.

$= \frac{1}{3} \lim\limits_{x\to\infty} x(x+1)$

$= \infty$.

$\lim\limits_{x\to-\infty} y(x) = \lim\limits_{x\to-\infty} \dfrac{x(x+1)}{3} = \infty$.

(4) Vertical asymptotes : $x=-3$.

(5) $f(x) = (x^3+4x^2+3x)(3x+9)^{-1}$

$f'(x) = (x^3+4x^2+3x)(-1)(3x+9)^{-2}(3) + (3x^2+8x+3)(3x+9)^{-1}$

$= \dfrac{-3(x^3+4x^2+3x) + (3x+9)(3x^2+8x+3)}{(3x+9)^2}$

$f'(x) = 0 \Leftrightarrow -3(x^3+4x^2+3x) + (3x+9)(3x^2+8x+3) = 0$

$\hookrightarrow$ simplifies to

$6x^3+39x^2+72x+27=0$.

$x = -3.,\ x = -\frac{4}{3}$.

Split Domain into 3 parts: $(-\infty,-3)$, $(-3,-4/3)$, $(-4/3,\infty)$

wk9

x	-4	-3	-2	$-4/3$	0
$f'(x)$	-21 <0	0	-9 <0	0	27

$\downarrow$ minimum pt.

(6)

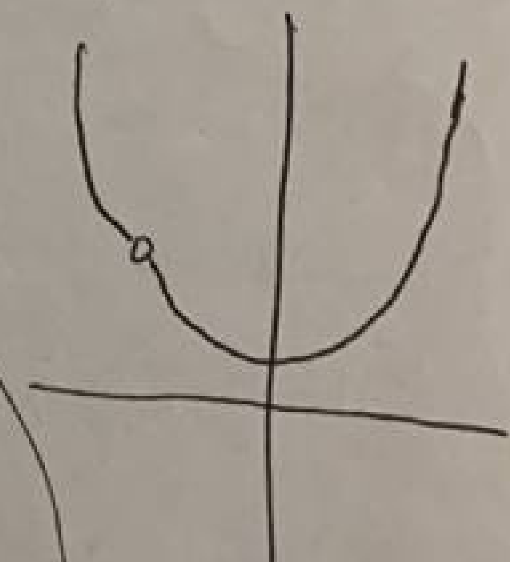
$y(x) = \dfrac{x^3+4x^2+3x}{3x+9}$

$= \frac{1}{3}(x^2+x)$.

$\therefore y'(x) = \frac{1}{3}(2x+1)$

$\Rightarrow y''(x) = \frac{2}{3} > 0$. $\}$ Always increasing.

$$\begin{array}{r}
\frac{1}{3}x^2 + \frac{1}{3}x \\
3x+9 \overline{)\, x^3+4x^2+3x} \\
-(x^3+3x^2) \\
\hline
x^2+3x \\
-(x^2+3x) \\
\hline
0.
\end{array}$$



(296) $y = \dfrac{2x+1}{x^2+6x+5}$.

WK9

x	-2	$x=\frac{13-\sqrt{221}}{2}$	0	$x=\frac{13+\sqrt{221}}{2}$	20
$f'(x)$					

Note that $\deg(2x+1) < \deg(x^2+6x+5)$.
So, horizontal asymptote is $y=0$.

① Domain: When $x^2+6x+5=0$, $y(x)$ is not defined.

$x^2+6x+5=0$.

$(x+5)(x+1)=0$

$x=-5$ or $x=-1$.

$\therefore$ Domain: $\mathbb{R}/\{-1,-5\}$.

x	-6	-5	-3	-2	-1.5	-1	0	1	2
$f'(x)$	<0	undefined	<0	0	>0	undefined	>0	$=0$	<0

② When $y=0$, $x=-\frac{1}{2}$. $(-\frac{1}{2},0) \leftarrow x$ intercept.
When $x=0$, $y=\frac{1}{5}$ $(0,\frac{1}{5}) \leftarrow y$ intercept

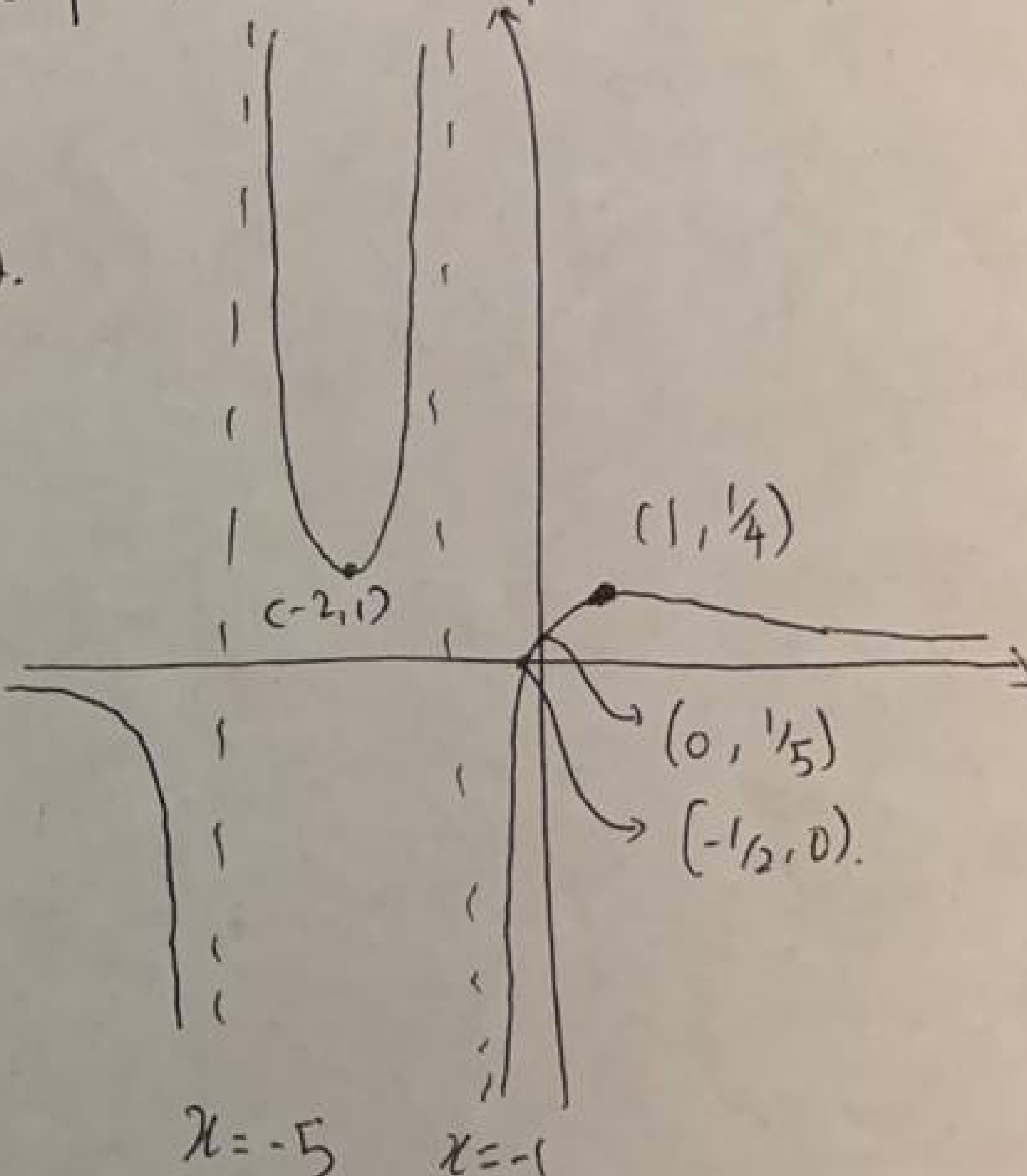
③ $$\lim_{x\to\infty} \frac{2x+1}{x^2+6x+5} = \lim_{x\to\infty} \frac{(\frac{1}{x^2})(2x+1)}{(\frac{1}{x^2})(x^2+6x+5)}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{1+\frac{6}{x}+\frac{5}{x^2}}$$

$$= \frac{0}{1} = 0.$$

$$\lim_{x\to-\infty} \frac{2x+1}{x^2+6x+5} = \frac{\lim_{x\to-\infty} \frac{2}{x}+\frac{1}{x^2}}{\lim_{x\to-\infty} 1+\frac{6}{x}+\frac{5}{x^2}}$$

$$= \frac{0}{1} = 0.$$



④ Vertical asymptotes are at the point $x=-1, x=-5$.

⑤ $y=(2x+1)(x^2+6x+5)^{-1}$

$y' = 2(x^2+6x+5)^{-1} + (2x+1)(-1)(x^2+6x+5)^{-2}(2x+6)$

$= \dfrac{2}{x^2+6x+5} - \dfrac{(2x+1)(2x+6)}{(x^2+6x+5)^2}$

$y'=0 \iff$ $2(x^2+6x+5)-(4x^2+14x+6)=0$

$2x^2+12x+10-4x^2+14x+6=0$

$-2x^2+26x+26=0$

$x^2-13x-13=0$.

$x=\dfrac{13+\sqrt{221}}{2}$ or $x=\dfrac{13-\sqrt{221}}{2}$

This is wrong.
turning points / critical pts are.
$x=-5, -2, -1, 1$.

Problem Solving Strategy

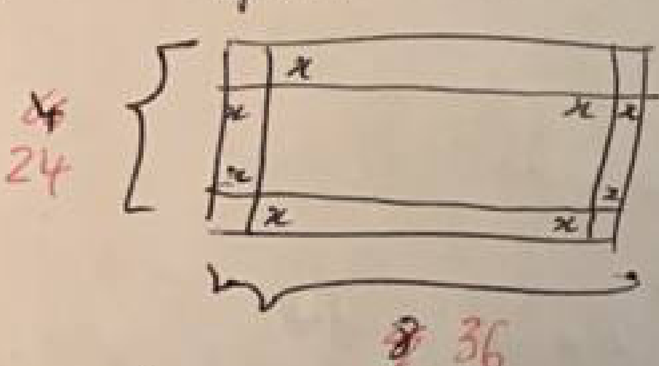
WK 10

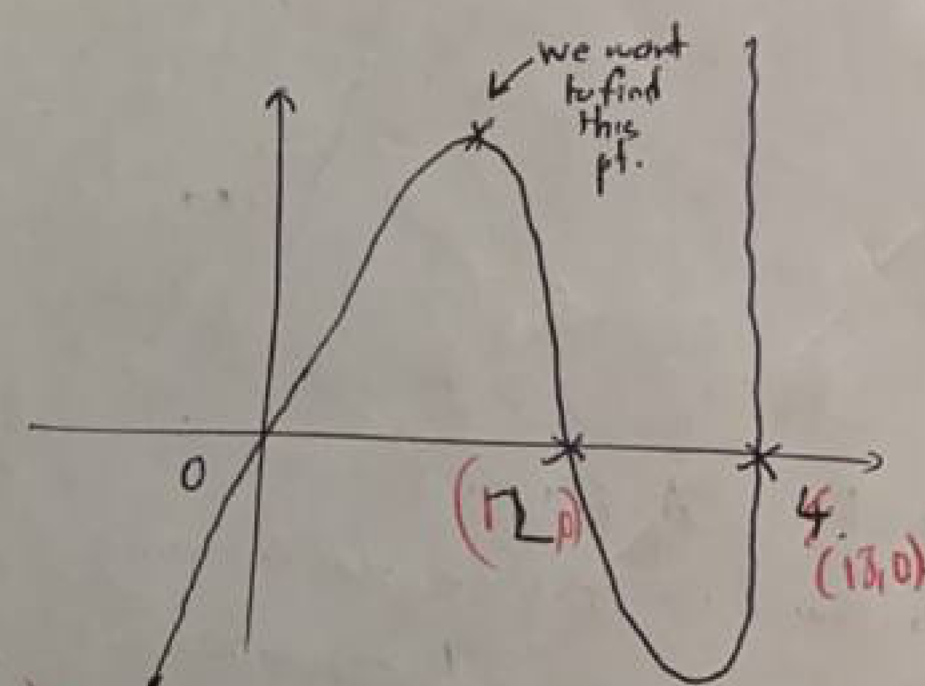
1. Introduce the variables.
2. Determine which quantity is to be maximised or minimized, and for what range of the other variables (if this can be determined at this time)
3. Write a formula for the quantity to be maximised or minized in terms of the variables. This formula may involve more than 1 variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximised/minimized as a function of 1 variable.
5. Identify the domain of the consideration for the function in step 4 based on the physical problem.
6. Locate the max/min value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Extreme Value Theorem: If f is a ctn function over the closed, bounded interval $[a,b]$, then $\exists$ a point $x \in [a,b]$ at which f has an absolute maximum over $[a,b]$ and $\exists$ a point in $[a,b]$ at which f has an absolute minimum over $[a,b]$.

Qn 316

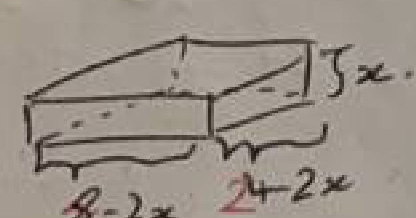
You are constructing a cardbox with dimensions ~~4~~ 24 cm × ~~8~~ 36 cm. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?





Note that $\cancel{0 < x < 2}$ $0 \le x \le 12$.

Volume of cuboid:



$$V(x) = (\cancel{8}\,36 - 2x)(\cancel{4}\,24 - 2x)(x) = (36-2x)(24-2x)x$$
$$= 4(4-x)(2-x)(x) \qquad = 4x^3 - 120x^2 + 864x$$
$$= 4(8-4x-2x+x^2)x$$
$$= 4x(x^2-6x+8).$$

We want to maximise $V(x)$.

Since V is ctn over the closed, bounded interval $[0, \cancel{2}\,12]$, V has a absolute min/max in $[0,2]$.

We find the critical pt. $V(x) = 4x^3 - \cancel{24}\,120x^2 + \cancel{32}\,864x$.

$$V'(x) = 12x^2 \cancel{- 48x + 32}. \qquad 12x^2 - 240x + 864 = 0$$

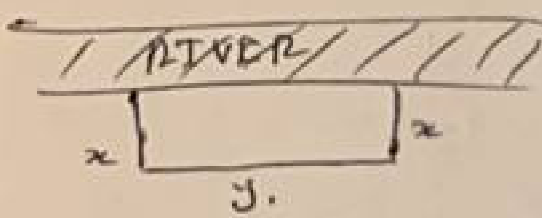
Solving for $V'(x) = 0$, $\quad x^2 - 20x + 72 = 0$

$$\cancel{12x^2 - 48x + 32 = 0}$$
$$\cancel{3x^2 - 12x + 8 = 0}$$

$$x = 10 \pm 2\sqrt{7} \quad (\text{rej } 10 + 2\sqrt{7})$$

$\therefore$ Vol is maximised when $x = 10 - 2\sqrt{7}$. Then $V(10-2\sqrt{7}) = 1825$#

(320) You have ~~800~~ 600 ft of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?



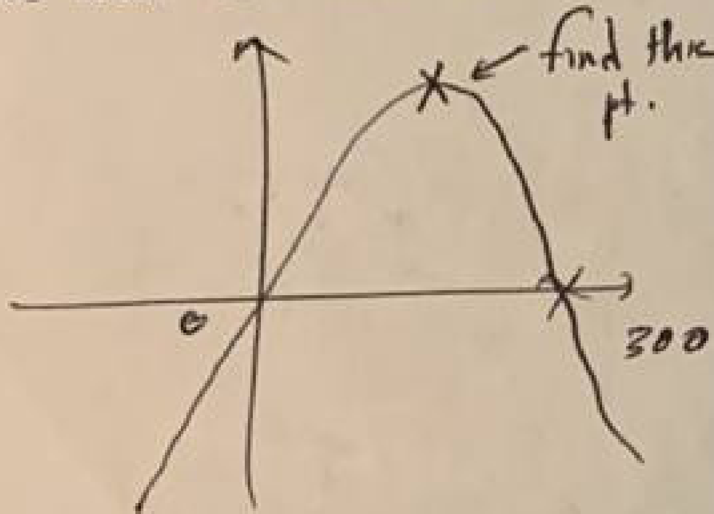
$\therefore 2x + y = \cancel{800}\ 600$.

We want to maximize Area of pen, $A(x,y) = xy$.

From $2x + y = 600$, we have ~~$A = xy$~~ $y = 600 - 2x$.

$\therefore A = xy = x(600 - 2x)$

We want to maximize A.



Note that $A = [0, 300]$.

So, we can use ~~extreme~~ extreme value theorem.

$A(x) = 600x - 2x^2$

$A'(x) = 600 - 4x$.

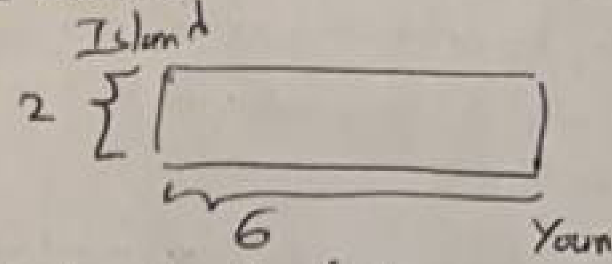
$\Rightarrow$ Solving for critical point,

$A'(x) = 600 - 4x = 0$

$\Rightarrow 4x = 600$

$x = 150m$.

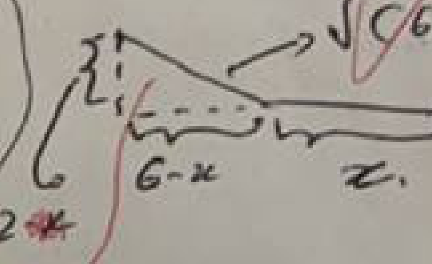
(326) You can run at a speed of ~~6mph~~ 8 mph and swim at ~~4~~ 3 mph, and are located on the shore, ~~4~~ 6 miles east of an island that is 2 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



Let x be the distance ran west.

$0 \le x \le 6$.

Then, time taken



~~$\sqrt{(6-x)^2 + (2-x)^2}$~~

~~$= \sqrt{36 - 12x + x^2 + 4 - 4x + 2x}$~~

~~$= \sqrt{40 - 16x + 3x^2}$~~

Distance of slanted line

$= \sqrt{(6-x)^2 + 2^2}$

$= \sqrt{(6-x)^2 + 4}$

Time taken $= \dfrac{x}{8} + \dfrac{\sqrt{(6-x)^2 + 4}}{3}$
$T(x)$

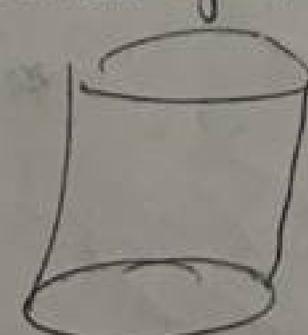
Remember $0 \le x \le 6$.

Since $T(x)$ is defined over a closed, bounded interval, $\exists$ a maximum/min.

$T'(x) = \dfrac{1}{8} - \dfrac{1}{2}\dfrac{[(6-x)^2 + 4]^{-1/2}}{3} \cdot 2(6-x) = \dfrac{1}{8} - \dfrac{(6-x)}{3\sqrt{(6-x)^2 + 4}}$

$\therefore x = 6 - \dfrac{6}{\sqrt{55}} \Rightarrow T(x) = T(6 - 6/\sqrt{55}) = 1.368h$.

Qn 341: Find the volume of the largest ~~cone~~ right circular cylinder that fits in a sphere of radius 2.



$h/2$, where h is height of cylinder

radius of cylinder

$(h/2)^2 + R^2 = 4$

$\therefore R^2 = 4 - h^2/4$.

Volume of cylinder $= \pi R^2 h$.

$= \pi\left(4 - \dfrac{h^2}{4}\right)h$.

$= \pi\left(4h - \dfrac{h^3}{4}\right)$

$\dfrac{dV}{dh} = \pi\left(4 - \dfrac{3}{4}h^2\right)$

$\dfrac{dV}{dh} = 0 \Rightarrow h = \dfrac{4}{\sqrt{3}} \Rightarrow \text{Volume} = \dfrac{32}{3\sqrt{3}}\pi$.

• L'Hôpital's Rule ← Using derivatives to calculate limits of a specific form

↳ Indeterminate forms.

WK10

Indeterminant forms → Type ①: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0}$

→ type ②: $\frac{\infty}{\infty}$.

Thm: L'Hôpital Rule (case $0/0$): Suppose f and g are differentiable functions over an open interval containing a, except possibly at.
If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0 \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Thm: L'Hôpital Rule (case ∞/∞): Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Other indeterminant forms: ③ $0 \cdot \infty$ (So, we want to evaluate $\lim_{x\to a} f(x)g(x)$, where $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$)

What we do in this case: $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} g(x) \div \frac{1}{f(x)}$

$$= \lim_{x\to a} g(x) \Big/ \left(\frac{1}{f(x)}\right)$$

④ $\infty - \infty$.

⑤ 0^0, ∞^0 and 1^∞. Evaluate $\lim_{x\to a} f(x)^{g(x)}$.

↳ How to solve this??

Let $y = f(x)^{g(x)}$

$\ln(y) = \ln[f(x)^{g(x)}]$

$= g(x)\ln f(x)$.

Calculate limit $g(x)\ln(f(x))$.

Suppose $g(x)\ln(f(x)) = L$.

$\Rightarrow \lim_{x\to a} \ln(y) = L$.

Since the natural logarithm is continuous, $\lim_{x\to a} \ln(y) = \ln\left(\lim_{x\to a} y\right) = L$.

$$\therefore \lim_{y\to a} = e^L.$$

MK 10

371 $\lim_{x\to\pi} \frac{x-\pi}{\sin(x)}$.

Note that $\lim_{x\to\pi} (x-\pi) = 0$

$\lim_{x\to\pi} \sin(x) = 0$.

$\therefore$ We have an indeterminant form.

Let $f(x) = x-\pi$

$g(x) = \sin(x)$

Then, $f'(x) = 1$

$g'(x) = \cos(x)$.

Then, $\lim_{x\to\pi} 1 = 1$

$\lim_{x\to\pi} \cos(x) = \cos(\pi) = -1$

$\therefore \lim_{x\to\pi} \frac{x-\pi}{\sin(x)} = \lim_{x\to\pi} \frac{1}{\cos(x)} = -1$.

374 $\lim_{x\to 0} \frac{(1+x)^n - 1 - nx}{x^2}$

Let $f(x) = (1+x)^n - 1 - nx$

$g(x) = x^2$

$\lim_{x\to 0} f(x) = (1+0)^n - 1 - 0 = 0$

$\lim_{x\to 0} g(x) = 0^2 = 0$.

$\therefore$ We have an indeterminant form.

376 $\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$.

$f(x) = \sqrt{1+x} - \sqrt{1-x}$

$g(x) = x$.

$\lim_{x\to 0} f(x) = \sqrt{1} - \sqrt{1} = 0$

$\lim_{x\to 0} g(x) = 0$.

So, we have the indeterminant form $\frac{0}{0}$.

$f'(x) = \frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}(1-x)^{-1/2}(-1)$

$= \frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}$.

$g'(x) = 1$.

$\therefore \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} f'(x)$

$= \lim_{x\to 0} \frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1+x}}$

$= 1$ #.

38? $\lim_{x\to\infty} x^2 e^{-x}$.

Let $f(x) = x^2$

$g(x) = e^{-x}$.

Then, $\lim_{x\to\infty} x^2 = \infty$

$\lim_{x\to\infty} e^{-x} = 0$.

$\therefore$ We have the indeterminant form $0\cdot\infty$.

$x^2 e^{-x} = \frac{x^2}{e^x}$

Then, $\lim_{x\to\infty} e^x = \infty$.

$\therefore$ We can switch the indeterminant form to $\frac{\infty}{\infty}$.

$f'(x) = 2x$

$g'(x) = -e^{-x}$.

$\lim_{x\to\infty} f'(x) = \infty$.

$\lim_{x\to\infty} g'(x) = 0$.

$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \frac{\infty}{0}$.

So, apply L Hopital Rule again,

$f''(x) = 2$

$g''(x) = e^{-x}$

$\therefore \lim_{x\to\infty} \frac{2}{e^x} = 0$.

395 $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x$.

Let $f(x) = 1-\frac{1}{x}$, $g(x) = x$

$\lim_{x\to\infty} f(x) = 1$ $\lim_{x\to\infty} g(x) = \infty$.

So, we have the indeterminant form, 1^∞.

Let $y = f(x)^{g(x)}$

$\ln(y) = g(x)\ln f(x)$.

$g(x)\ln f(x) = x\ln\left(1-\frac{1}{x}\right)$.

$\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right)$

$= \left(\lim_{x\to\infty} x\right)\ln\left(\lim_{x\to\infty} 1-\frac{1}{x}\right)$

$= \infty\cdot 1 = \infty$.

So, $\lim_{x\to\infty} \ln(y) = \infty$.

$\ln\left(\lim_{x\to\infty} y\right) = \infty$

$\therefore \lim_{x\to y} y = e^\infty = \infty$.

Antiderivatives

WK 11

→ Given a function $F(x)$, we have seen how to calculate the derivative $F'(x)$.

→ Now, we want to reverse this process around. Given a function $f(x)$, how do we find a function with the derivative f?

→ Motivation: Comes ~~up with~~ from classical mechanics.

Definition: A Function F is an antiderivative of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

⇒ Notes on the constants: If F is an antiderivative of f over an interval I ⇒ $F(x)+C$ is also an antiderivative of f over I.

Examples of antiderivatives: $f(x) = 3x^2, \frac{1}{x}, \cos x, e^x$.

Indefinite Integrals

• We look at the formal notation used to represent antiderivatives.

→ Given a function f, we use the notation $f'(x)$ to denote the derivative of f.

→ If F is the antiderivative of f (i.e. $F'(x) = f(x)$), we say that $F(x)+C$ is the most general antiderivative of f and write

called the indefinite integral of f.

$$\int f(x)\,dx = F(x) + C.$$

$\int$ called the integral sign.

Definition: Given a function f, the indefinite integral of f, denoted $\int f(x)\,dx$, is the most general antiderivative of f. If F is an antiderivative of f ⇒ $\int f(x)\,dx = F(x)+C$.

The expression $f(x)$ is ~~#~~ called the integrand and the variable x is the variable of integration.

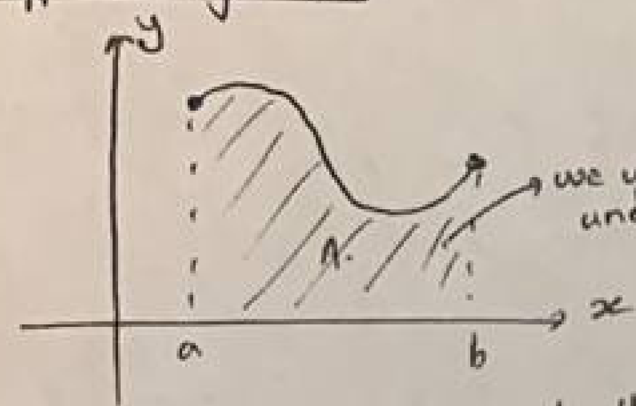
Theorem:

1. For $n \neq -1$, $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$.
2. $\int k\,dx = kx + C$
3. $\int \frac{1}{x}\,dx = \ln|x| + C$.
4. $\int e^x\,dx = e^x + C$.
5. $\int \cos x\,dx = \sin(x) + C$.
6. $\int \sin(x)\,dx = -\cos(x) + C$.
7. $\int \sec^2(x)\,dx = \tan(x) + C$.
8. $\int \csc x \cot x\,dx = -\csc(x) + C$.
9. $\int \sec x \tan x\,dx = \sec(x) + C$.
10. $\int \csc^2 x\,dx = -\cot(x) + C$.
11. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$.
12. $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$.
13. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C$.

→ Properties of Integrals: $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int kf\,dx = k\int f(x)\,dx$.

Approximating Areas



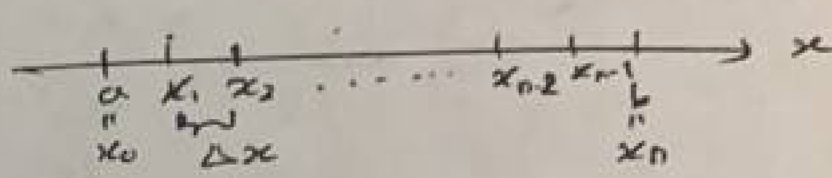
we want to find area under this graph.

UC!!

$\Rightarrow$ We will do this geometrically: By dividing the region into many small shapes that have known Area formulas: (For example rectangles).

Step ①: Divide the interval $[a,b]$ into n intervals of equal width, $\frac{b-a}{n}$. We do this by ~~eq~~ selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a$ and $x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

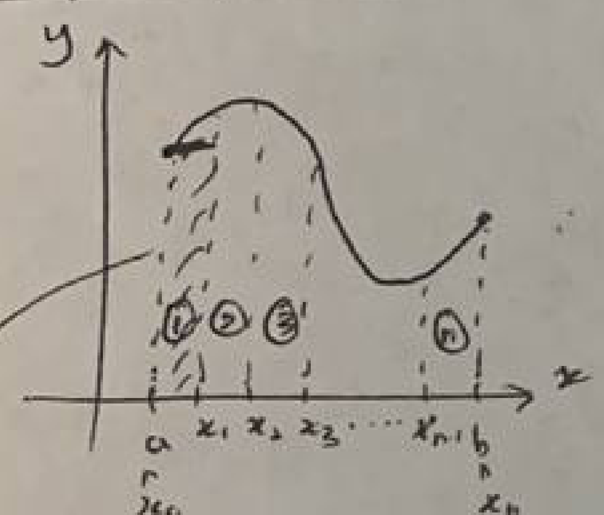


for $i = 1, 2, 3, \cdots, n$.

We denote the width of each subinterval with the notation Δx. So, $\Delta x = \frac{b-a}{n}$.

$\Rightarrow$ $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called the partition of $[a,b]$.

Left-Endpoint Approximation



$\rightarrow$ Take the left end of each subinterval to approximate the area.

$\rightarrow$ Area of rect ①: $f(x_0)\Delta x$.

②: $f(x_1)\Delta x$

$\vdots$

ⓝ : $f(x_{n-1})\Delta x$.

$\therefore$ Approximation of Area under the graph $= f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$

$$= \sum_{i=0}^{n-1} f(x_i)\Delta i$$

Right hand Approximation

$\rightarrow$ Area of rectangle ①: $f(x_1)\Delta x$

②: $f(x_2)\Delta x$

③: $f(x_3)\Delta x$

$\vdots$

ⓝ: $f(x_n)\Delta x$.

$\therefore$ Approximation of Area under the graph $= \sum_{i=1}^{n} f(x_i)\Delta x$.

Riemann Sum $= \sum_{i=1}^{n} f(x_i^*)\Delta x$.

Area under the curve $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$.

- Application of Antiderivatives: Solving Differential equations

→ A Differential Equation is an equation that relates an unknown function and one or more of its derivatives. The eqn

$$\frac{dy}{dx} = f(x)$$

WK 11

is a simple example of a differential Equation.

Solving this equation means finding a function y with derivative f.

→ Thus, the solutions of DEs are antiderivatives.

=) Remember that antiderivatives come as a family of soln.

→ So, the problem of finding a function y that satisfies a DE with the additional condition

$$y(x_0) = y_0$$

is an example of an initial value problem.

Qn 466 Show that F(x) are antiderivatives of f(x).

$F(x) = x^2 + 4x + 1$, $f(x) = 2x + 4$.

By definition of antiderivatives, $F'(x) = \frac{d}{dx} F(x)$

$= \frac{d}{dx}(x^2+4x+1)$

$= 2x + 4.$

Qn 470 For the following exercises, find the antiderivatives of the function.

→ $f(x) = \frac{1}{x^2} + x.$

To find the antiderivative, we calculate $\int f(x)\,dx$.

So, $\int \frac{1}{x^2} + x\,dx = \int \frac{1}{x^2}\,dx + \int x\,dx$

$= \int x^{-2}\,dx + \int x\,dx$

$= \frac{x^{-2+1}}{(-2+1)} + \frac{x^{1+1}}{2} + C$

$= -x^{-1} + \frac{1}{2}x^2 + C$ #.

Qn 477.

→ $f(x) = (\sqrt{x})^3$

$= x^{3/2}.$

$\int f(x)\,dx = \int x^{3/2}\,dx$

$= \frac{x^{3/2+1}}{(3/2+1)} + C$

$= \frac{2}{5}x^{5/2} + C$ #.

Qn 493 For the following Qn, evaluate the integral

$\int \frac{3x^2+2}{x^2}\,dx = \int \frac{3x^2}{x^2} + \frac{2}{x^2}\,dx$

$= \int 3 + \frac{2}{x^2}\,dx = \int 3 + 2x^{-2}\,dx$

$= \int 3\,dx + 2\int x^{-2}\,dx$

$= 3x + \frac{x^{-2+1}}{(-2+1)} + C$

$= 3x + \frac{x^{-1}}{-1} + C.$

Qn 499 : Solve the initial value problem

$f'(x) = x^{-3}$, $f(1) = 1$

⇓

$\frac{dy}{dx} = x^{-3}$, $f(1) = 1$

$\int \frac{dy}{dx}\,dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{(-3+1)} + C = \frac{x^{-2}}{-2} +$

$\therefore y(x) = -\frac{1}{2}x^{-2} + C.$

=) Since $f(1) = 1$, $y(1) = 1$.

ie $-\frac{1}{2}(1)^{-2} + C = 1$ $\therefore -\frac{1}{2} + C = 1$

$\therefore C = 3/2.$

Qn 511: You are merging onto the freeway, accelerating at a constant rate of 12 ft/sec². How long does it take to reach merging speed at 60 miles/hr?

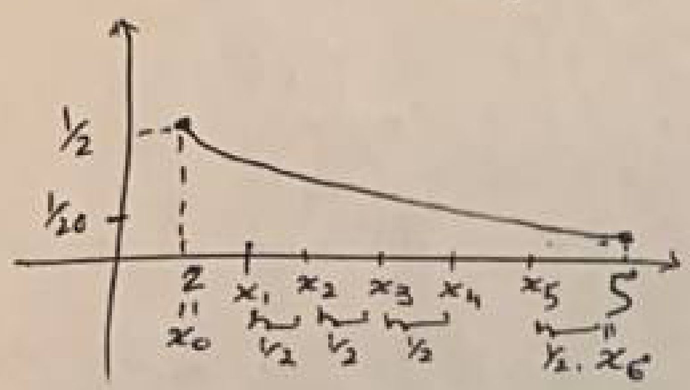
$a(t) = +12\,ft/sec^2.$

speed $= \int a(t)\,dt = 12t + C.$

$s(t) = 12t + c.$

when $t = 0$, $s(0) = 12(0) + C = 12.$

Qn14. L_6 for $f(x)=\frac{1}{x(x-1)}$, on $[2,5]$.



$\Delta x = \frac{5-2}{6} = 1/2.$

$f(x_0)\Delta x = f(2)(1/2) = \frac{1}{4}.$

$f(x_1)\,\Delta x = f(2.5)(1/2) = \frac{1}{(5/2)(3/2)}\,(1/2)$

$= \frac{4}{15}\times 1/2 = 2/15.$

$f(x_2)\Delta x = f(3)(1/2) = \frac{1}{3\times 2}\left(\frac{1}{2}\right) = 1/12.$

$f(x_3)\Delta x = f(3.5)(1/2) = \frac{1}{7/2\times 5/2}\times 1/2 = 1/35$

$f(x_4)\Delta x = f(4)(1/2) = \frac{1}{12}\times 1/2 = \frac{2}{24}$

$f(x_5)\Delta x = f(4.5)(1/2) = \frac{1}{(9/2)(7/2)}\times 1/2 = 2/63.$

$\therefore L_6 = \sum_{i=0}^{5} f(x_i)\Delta x = \frac{1}{4}+\frac{2}{15}+1/12+1/35+2/24+2/63.$

Qn 15: R_6 for the same function.

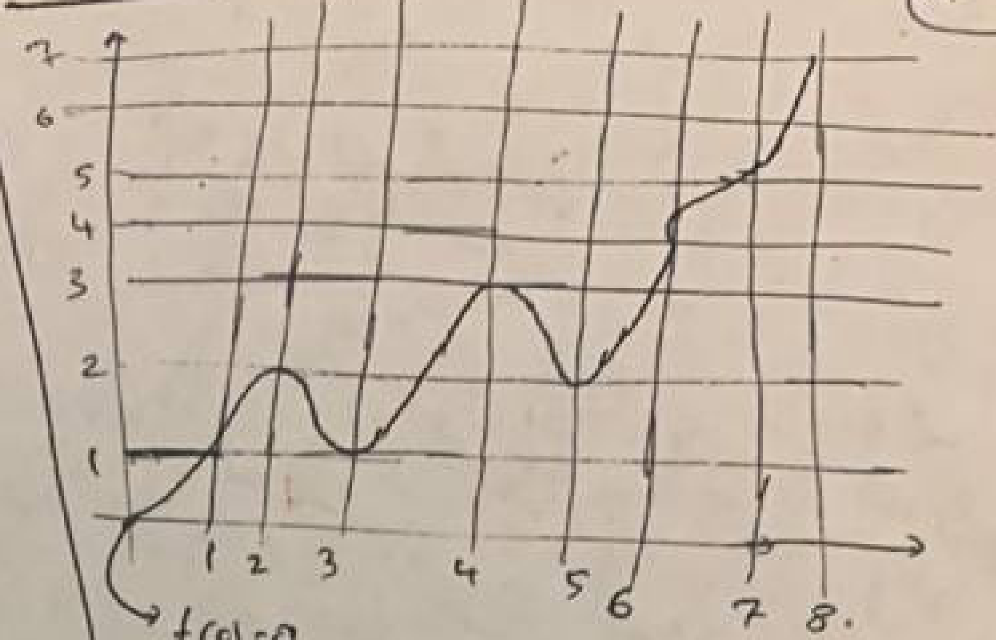
$f(x_1)\Delta x,\ f(x_2)\Delta x,\ f(x_3)\Delta x,\ f(x_4)\Delta x,\ f(x_5)\Delta x$ has already been calculated.

$\Rightarrow f(x_6)\Delta x = f(5)\Delta x = \frac{1}{20}\cdot 1/2 = 1/40.$

$\therefore R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = \frac{2}{15}+1/12+1/35+2/24+2/63+1/40.$

Qn44 Compute L_8.

11



$f(0)=0$
$f(1)=1$
$f(2)=2$
$f(3)=1$
$f(4)=3$
$f(5)=2$
$f(6)=4$
$f(7)=5.$

$\Delta x = 1$

$\therefore L_8 = 0\cdot1+1\cdot1+2\cdot1+1\cdot1+3\cdot1+2\cdot1+4\cdot1+5\cdot1$

$= 0+1+2+1+3+2+4+5$

$= 18$ #.

WK12

- Given a curve $f(x)$, we defined the area under the curve in terms of Riemann Sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

→ For this method to work, we require $f(x)$ to be continuous and non-negative.

→ So, we will find a new concept that generalises the notion of area under the curve (we lift the requirements that $f(x)$ be continuous and non-negative).

Definition: If $f(x)$ is a function defined on an interval $[a,b]$, the definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

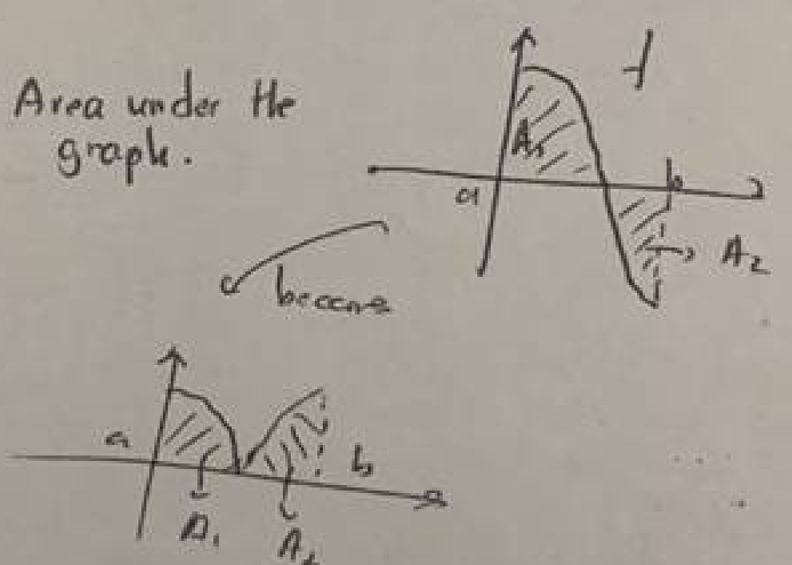
provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a,b]$, or is an integrable function.

Evaluating Definite Integrals

- Calculating Definite integrals using riemann sums is tedious.
- So, at times, we can rely on the fact that definite integrals represent the area under the curve, and we can calculate definite integrals using geometric formulas to calculate the area.

Net Signed Area

- (Area above the x-axis) − (Area below the y axis) = Area under the graph.
  ↑ Net signed area.



Area of f from a to b $= A_1 - A_2$.

- total Area of $f(x)$ $= \int_a^b |f(x)|\,dx = A_1 + A_2$

Properties of Definite Integral:

① $\int_a^a f(x)\,dx = 0.$

② $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

③ $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

④ $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx.$

⑤ $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$

WK12

Qn 61 : Express the limit as an integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(5(x_i^*)^2 - 3(x_i^*)^3\right) \Delta x \quad \text{over} \quad \underline{\underline{[0,2]}}$$

$\downarrow$ gives us $[a,b]$.

We need to write this as $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$.

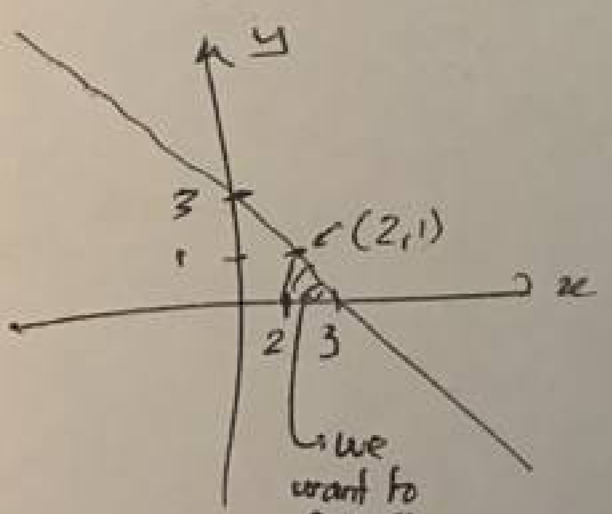
$\therefore \int_0^2 5x^2 - 3x^3 \, dx$ #.

(77) Evaluate the ~~area~~ integral using the area formula.

$$\int_{2}^{3} (3-x)\,dx.$$

$(3-x)$ — this is our function

$\int_2^3$ — this gives us the region.



$\rightarrow$ So, this is a triangle.

$\therefore$ Area $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ #

(91) Suppose that $\int_0^2 f(x)\,dx = -3$ and $\int_0^2 g(x)\,dx = 2$. Then, compute the integral.

Also $\int_0^4 f(x)\,dx = 5$, $\int_0^4 g(x)\,dx = -1$.

Compute $\int_2^4 f(x) - g(x)\,dx$.

$$\int_2^4 f(x) - g(x)\,dx = \int_2^4 f(x)\,dx - \int_2^4 g(x)\,dx$$

$$= \int_0^4 f(x)\,dx - \int_0^2 f(x)\,dx - \left[\int_0^4 g(x)\,dx - \int_0^2 g(x)\,dx\right]$$

$$= 5 - (-3) - \left[(-1) - 2\right]$$

$$= 8 - (-3) = 11 \text{ #}$$

- More tools to calculate definite Integrals: The Fundamental theorem of Calculus.

→ Mean value theorem for integrals: If $f(x)$ is continuous over ~~the~~ an interval $[a,b]$ ⇒ there is at least one point $c \in [a,b]$ such that $f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}$.

Also written as:

$$(b-a)f(c) = \int_a^b f(x)\,dx.$$

- Fundamental Theorem of Calculus I: If $f(x)$ is continuous over an interval $[a,b]$, and the function $F(x)$ is defined by $F(x) = \int_a^x f(t)\,dt$ ⇒ $F'(x) = f(x)$ over $[a,b]$.

- Fundamental theorem of Calculus II: If f is continuous over the interval $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ ⇒ $\int_a^b f(x)\,dx = F(b) - F(a)$.

148 Using FToC I, Find each derivative.

Given: $\frac{d}{dx}\int_1^x e^{-t^2}\,dt$.

Let $F(x) = \int_1^x e^{-t^2}\,dt$.

∴ We need to find $\frac{d}{dx}F(x) = F'(x)$ } by FToC I

$= f(x)$

$= e^{-x^2}$

156 $\frac{d}{dx}\underbrace{\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt}$

Let this by $G(x)$.

$$G(x) = \int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt.$$

Let $u = \sqrt{x}$.

So, we have $\underbrace{G(x)}_{G(u(x))} = \int_1^{u(x)} \frac{t^2}{1+t^4}\,dt.$

∴ $\frac{dG}{dx} = \frac{dG}{du}\cdot\frac{du}{dx}$. (Need to use Chain rule).

$$\frac{dG}{dx} = \frac{dG}{du}\cdot\frac{du}{dx} = \frac{u(x)^2}{1+u(x)^4}\left(\frac{du}{dx}\right) = \frac{x}{1+x^2}\left(\frac{1}{2}x^{-1/2}\right).$$

WK 12

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175 Evaluate the definite integral using FToC 2:

WK12

$$\int_0^1 x^{99}\,dx.$$

We know that antiderivative of $x^{99} = \frac{x^{100}}{100}$

So, $\int_0^1 x^{99}\,dx = \left[\frac{x^{100}}{100}\right]_0^1$

$$= \frac{1^{100}}{100} - \frac{0^{100}}{100}$$

$$= \frac{1}{100} \#$$

WK 13

• The Net change theorem considers the integral of a rate of change: It says that when a quantity changes, ~~the new~~ value equals the initial value plus the integral of the rate of change of the quantity.

Net change theorem: The new value of a changing quantity equals the ~~end~~ initial value plus the integral of the rate of change:

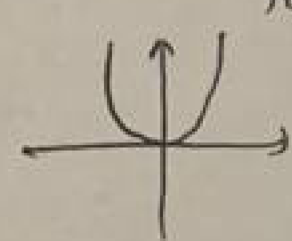
$$F(b) = F(a) + \int_a^b F'(x)\,dx \quad \text{or}$$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$ This is the Fundamental theorem of Calculus I rearranged.

Definition of Even function: An even function is a function in which $f(-x) = f(x), \forall x \in$ Domain.

E.g: $y = x^2 = f(x)$ $\quad f(x) = x^2$

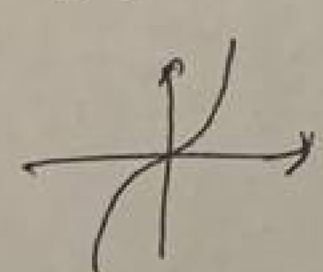
$f(-x) = (-x)^2 = x^2$.



Definition of Odd function: An odd function is one in which $f(-x) = -f(x), \forall x \in$ Domain.

E.g: $y = x^3 = f(x)$. $\quad f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$ #.



Rule: For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Qn 2: An object moves along a straight line with velocity $v(t) = 2t^2 - 3t$ m/sec.

(a) Find the displacement of the object over the time interval $[0,5]$.

(b) Find the total distance the object travels over the time interval $[0,5]$.

(a) $\int_0^5 v(t)\,dt = \int_0^5 2t^2 - 3t\,dt$

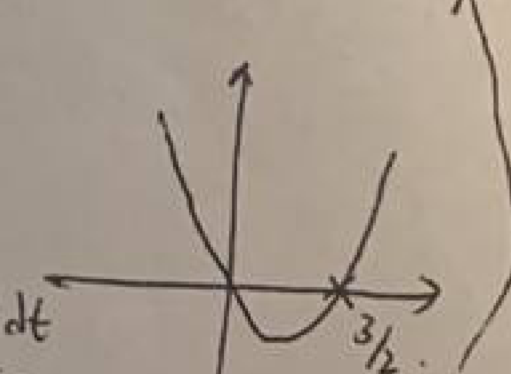
$= \left[\frac{2t^3}{3} - \frac{3t^2}{2}\right]_0^5$

$= \frac{2(5)^3}{3} - \frac{3(5)^2}{2}$

(b) $\int_0^5 |v(t)|\,dt$

$= \int_0^{3/2} -v(t)\,dt + \int_{3/2}^5 v(t)\,dt$

$= \int_0^{3/2} 3t - 2t^2\,dt + \int_{3/2}^{5} 2t^2 - 3t\,dt$



$= \left[\frac{3t^2}{2} - \frac{2t^3}{3}\right]_0^{3/2} + \left[\frac{2t^3}{3} - \frac{3t^2}{2}\right]_{3/2}^{5}$

Qn 265

$$\int \frac{x}{\sqrt{x^2+1}}\,dx; \quad u = x^2+1.$$

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx.$$

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{x^2+1}}\,(x\,dx)$$

$$= \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\right) du$$

$$= \frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{1/2}}{(1/2)} + C = u^{1/2} + C.$$

$$= \sqrt{x^2+1} + C.$$

Qn 275 $\int \cos^3\theta \sin\theta \, d\theta$. (HK 陈)

$$= \int (\cos\theta)^3 \sin\theta \, d\theta.$$

Note that $\frac{d}{d\theta} \cos\theta = -\sin\theta$.

$\therefore$ Let $u = \cos\theta$.

$$\frac{du}{d\theta} = -\sin\theta.$$

$$\Rightarrow -du = \sin\theta\, d\theta.$$

$$\therefore \int (\cos\theta)^3 \sin\theta \, d\theta$$

$$= \int u^3 (-1)\, du$$

$$= -\int u^3\,du = -\frac{u^4}{4} + C$$

$$= -\frac{\cos^4\theta}{4} + C.$$

Integrals of Exponential functions:

① $\int e^x\,dx = e^x + C$ ② $\int a^x\,dx = \frac{a^x}{\ln(a)} + C.$ ③ $\int x^{-1}\,dx = \ln|x| + C$ ④ $\int \ln(x)\,dx = x\ln(x) - x + C$ $= x(\ln(x)-1) + C.$

⑤ $\int \log_a(x)\,dx = \frac{x}{\ln(a)}(\ln(x) - 1) + C.$

Qn 321. Evaluate $\int e^{-3x}\,dx$.

Let $u = -3x$. $\Rightarrow \frac{du}{dx} = -3$

$$\Rightarrow \frac{1}{-3}du = dx.$$

So, $\int e^{-3x}\,dx = \int e^u\left(-\frac{1}{3}\right)du = -\frac{1}{3}\int e^u\,du$

$$= -\frac{1}{3}e^u + C$$

$$= -\frac{1}{3}e^{-3x} + C.$$

Qn 323 $\int 3^{-x}\,dx$. Let $y = -x$. $\Rightarrow \frac{dy}{dx} = -1$

$$\Rightarrow -dy = dx.$$

$$\int 3^{-x}\,dx = \int 3^y(-1)\,dy = -\int 3^y\,dy = -\left[\frac{3^y}{\ln(3)} + C\right]$$

$$= -\frac{3^{-x}}{\ln(3)} + C \ \#$$

Substitution

Wk 12

→ The fundamental theorem of calculus gives us a method to evaluate integrals without using riemann sums. The drawback is that we need to be able to find the derivative, which is not always easy.

→ So, we have a new technique to find antiderivatives: Integration by Substitution.

→ This method helps us find antiderivatives when the integrand is the result of a chain rule derivation.

Theorem: Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\int f[g(x)]\,g'(x)\,dx = \int f(u)\,du \qquad \left(\begin{array}{l} \text{As } u = g(x) \\ \frac{du}{dx} = g'(x) \\ \Rightarrow du = g'(x)\,dx \end{array} \right)$$

$$= F(u) + C$$

$$= F(g(x)) + C.$$

Problem-Solving Strategy

① Look carefully at the integrand and select an expression $g(x)$ with the integrand set equal to u. Let's select $g(x)$ such that $g'(x)$ is also part of the integrand.

② Substitute $u = g(x)$ and $du = g'(x)dx$ into the integral.

③ We should be able to evaluate the integral with respect to u. If the integrand can't be evaluated we need to go back and select a different expression to use as u.

④ Evaluate the integral in terms of u.

⑤ Write the result in terms of x and the expression $g(x)$.

Substitution with definite integrals: Let $u = g(x)$, and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$.

Then, $$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Exercises: find the antiderivative using the indicated substitution.

Qn: (261). $\int (x+1)^4\,dx;\ u = x+1.$

$$\Rightarrow \frac{du}{dx} = 1.$$

$$\therefore du = dx.$$

$$\Rightarrow \int (x+1)^4\,dx = \int u^4\,du = \frac{u^5}{5} + C$$

$$= \frac{(x+1)^5}{5} + C.\ \#$$

Qn (263): $\int (2x-3)^7\,dx;\ u = 2x-3.$

~~Let~~ $\Rightarrow \frac{du}{dx} = 2. \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

$$\Rightarrow \int (2x-3)^{-7}\,dx = \int u^{-7}\left(\tfrac{1}{2}\right)du$$

$$= \frac{1}{2}\int u^{-7}\,du$$

$$= \frac{1}{2}\left[\frac{u^{-6}}{-6}\right] + C$$

$$= -\frac{1}{12}u^{-6} + C$$

$$= \frac{-(2x-3)^{-6}}{12}$$

(5) Let $C(x)$ denote the cost to produce x widgets. The marginal cost is given by $C'(x) = 2 - \frac{3}{1000}(x-4)^2$, for $x \geq 4$. If the cost of the first 4 widgets is \$500, find the cost to produce the first 24 widgets.

Soln.

Cost = Cost of first 4 + Cost to make 5^{th} to 24^{th} widget

$$= \$500 + \int_5^{24} C'(x)\,dx$$

$$= \$500 + \int_5^{24} 2 - \frac{3}{1000}(x-4)^2\,dx$$

$$= \$500 + \int_5^{24} 2\,dx - \frac{3}{1000}\int_5^{24} (x-4)^2\,dx$$

$$= 500 + \left[2x\right]_5^{24} - \frac{3}{1000}\left[\frac{(x-4)^3}{3}\right]_5^{24}$$

(6) An insect population is growing at a rate of $2^{t/8}$ insects / day. Find the population of the insect after 24 days assuming there are 100 insects at time $t=0$.

$$P'(t) = 2^{t/8}$$

$\to P(t)$ population of insect over time t.

$$\therefore P(24) = \int_0^{24} P'(t)\,dt$$

$$= \int_0^{24} 2^{t/8}\,dt.$$

File_001.png

③ An object moves along a straight line with velocity given by

WKB

$$v(t) = \begin{cases} t, & 0 \le t \le 2 \\ 4-t, & t \geq 2. \end{cases}$$

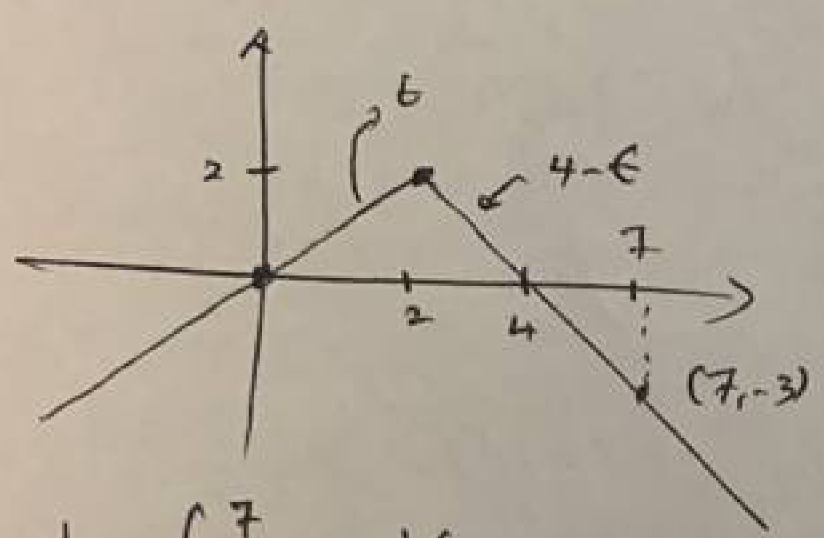
in ft/sec.

(a) Sketch the graph of $v(t)$ over the time interval $[0,7]$.

b) Find the displacement of the object over the time interval $[0,7]$ by looking at your graph and using the signed area interpretation of a definite integral.

ⓒ Find the total distance the object travels over the time interval $[0,7]$, again by using areas.

ⓐ



(b) Displacement $= \underbrace{\int_0^7 v(t)\,dt}_{\text{Area under the graph from } t=0 \text{ to } t=7}$.

So, using signed areas, $\int_0^7 v(t)\,dt = $ Area of Δ from 0 to 4 $-$ Area of Δ from 4 to 7

$$= \frac{1}{2}(4)(2) - \frac{1}{2}(3)(3)$$

$$= 4 - \frac{9}{2} = -\frac{1}{2}.$$

ⓒ $\frac{1}{2}(4)(2) + \frac{1}{2}(3)(3) = \frac{17}{2}.$

④ Water is flowing into a tank at the rate of $t - \sin(3\pi t/4)\ cm^3/min$. How much water flows into the tank over the time interval from 2 to 4 minutes.

$$\frac{dV}{dt} = t - \sin(3\pi t/4).$$

$\Rightarrow$ Amt of water $= \int_2^4 \frac{dV}{dt}\,dt = \int_2^4 t - \sin\left(\frac{3\pi t}{4}\right) dt$

$$= \left[\frac{t^2}{2} + \cos\left(\frac{3\pi t}{4}\right) \Big/ \left(\frac{3\pi}{4}\right)\right]_2^4$$

$$= \frac{16}{2} + \frac{4}{3\pi}\cos(12\pi/4) - \left[\frac{4}{2} + \frac{4}{3\pi}\cos(6\pi/4)\right]$$

File_006.png

WCIB

① $\int e^x \, dx = e^x + C$

② $\int a^x \, dx = \frac{a^x}{\ln(a)} + C.$

=> A common mistake when dealing with exponential ~~function~~ expressions is treating the exponent the same way we treat exponents in polynomial expressions.

We CANNOT use power rule for the exponent on e.

=> Use Substitution.

③ $\int x^{-1} \, dx = \ln|x| + C.$

④ $\int \ln|x| \, dx = x\ln x - x + C = x(\ln x - 1) + C.$

⑤ $\int \log_a x \, dx = \frac{x}{\ln(a)} (\ln(x) - 1) + C.$

Qns.

324 $\int \frac{1}{2x} \, dx = \frac{1}{2} \int \frac{1}{x} \, dx$

$= \frac{1}{2} \int x^{-1} \, dx = \frac{1}{2} \ln|x| + C.$

320 $\int e^{2x} \, dx.$

Let $u = 2x$.

$\Rightarrow x = \frac{u}{2}.$

$\Rightarrow \frac{dx}{du} = \frac{1}{2} \Rightarrow dx = \frac{1}{2} du.$

$\int e^{2x} \, dx = \int e^u \left(\frac{1}{2} du\right)$

$= \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C.$

$= \frac{1}{2} e^{2x} + C.$

330 $\int \frac{1}{x(\ln x)} \, dx = \int \frac{1}{\ln(x)} \left(\frac{1}{x}\right) dx.$

Let $u = \ln(x)$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}.$

$\Rightarrow du = \frac{1}{x} dx.$

So, $\int \frac{1}{x(\ln x)} dx = \int \frac{1}{\ln(x)} \left(\frac{1}{x} dx\right)$

$= \int \frac{1}{u} \, du$

$= \ln|u| + C$

$= \ln|\ln(x)| + C$ #.

337 $\int x^2 e^{-x^3} \, dx.$

Let $u = -x^3.$

$\Rightarrow \frac{du}{dx} = -3x^2$

$\Rightarrow$ ~~dx~~ $dx = \frac{1}{-3x^2} du.$

$\Rightarrow x^2 \, dx = -\frac{1}{3} du.$

$\int x^2 e^{-3x^3} dx = \int e^{-3x^2} (x^2 dx) = \int e^u \left(-\frac{1}{3}\right) du = -\frac{1}{3} e^{u},$

$= -\frac{1}{3} e^{-x^3} + C_1$

WK 13

① $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{|a|} + C.$

② $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$

③ $\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{|a|}\sec^{-1}\left(\frac{|u|}{a}\right) + C$

Qns

392 $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{|1|}\right)\right]_{-1/2}^{1/2}$

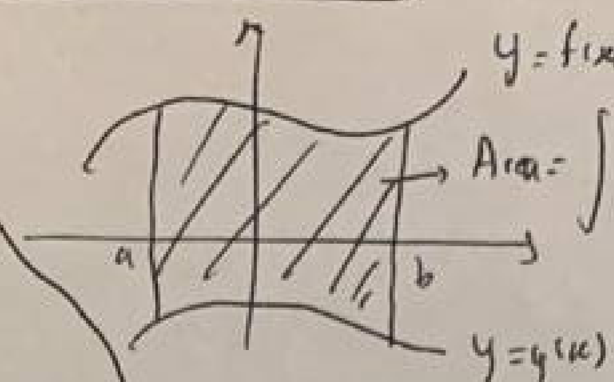
$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

~~394~~

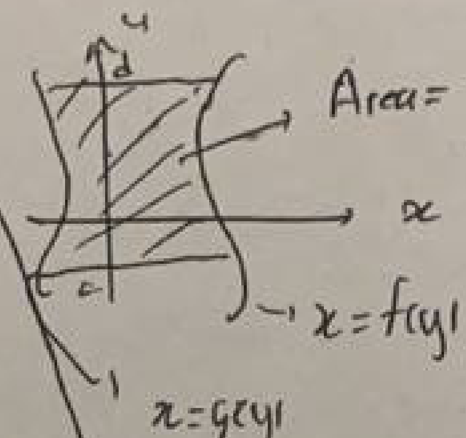
397 $\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{3^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + C.$

400 $\int \frac{1}{25+16x^2}\,dx = \int \frac{1}{5^2+(4x)^2}\,dx = \frac{1}{5}\tan^{-1}\left(\frac{4x}{5}\right) + C$ #.

Area between Curves

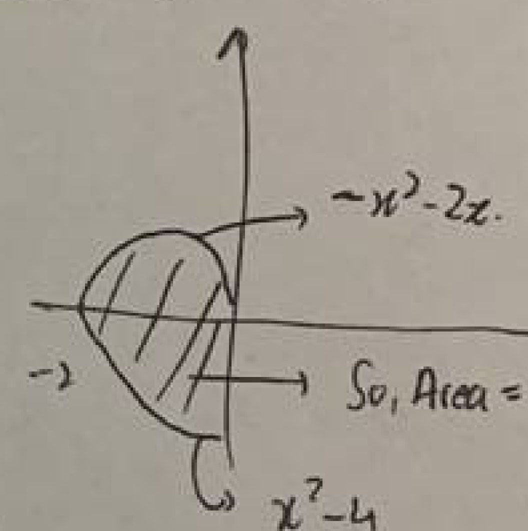


Area $= \int_a^b |f(x)-g(x)|\,dx.$



Area $= \int_c^d |f(y)-g(y)|\,dy.$

Example $y = x^2-4,\ y = -x^2-2x$



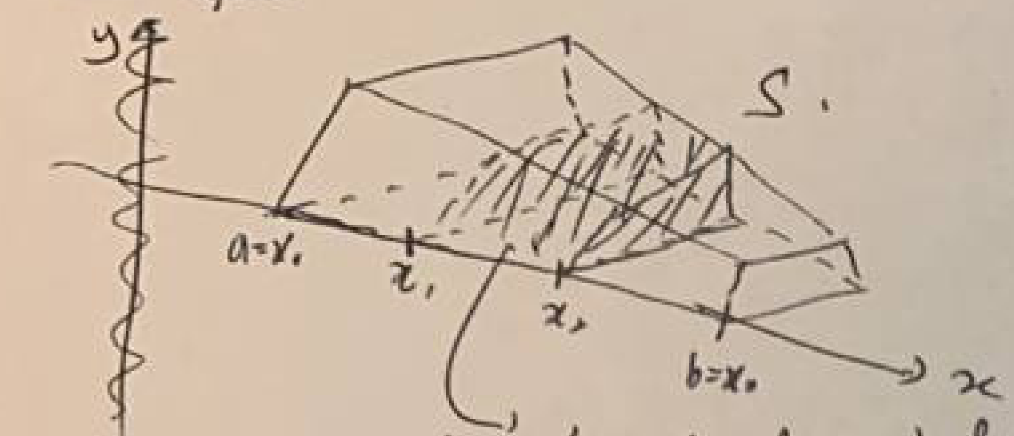
So, Area $= \int_{-2}^{0} |(-x^2-2x) - (x^2-4)|.$

Volume of Cylinder / Determining Volume

- If a solid doesn't have a constant cross-section, we ~~be~~ may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid.

Intuition: We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding these estimated volumes together.

WK 14



Area of solid $\cong \Delta x \cdot$ Area of one side of shape

$\therefore$ Volume of the slice can be estimated by $V(S_i) \cong A(x_i^*)\,\Delta x$.

$$\therefore V \cong \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

$$\Rightarrow V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx.$$

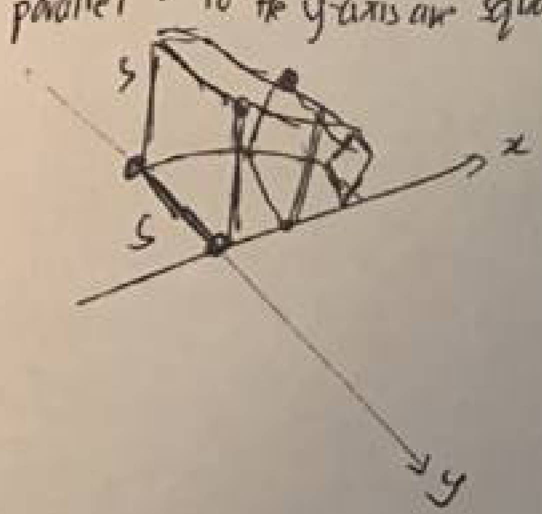
Problem-Solving Strategy: Finding Volume by Slicing.

1) Examine the solid and determine the shape of a cross section of the solid. It is often helpful to draw a picture if one is not provided.

2) Determine a formula for the area of the cross-section.

3) Integrate the area formula over the appropriate interval to get the volume.

(70) The base is the region under the parabola $y=1-x^2$ in the first quadrant. Slices perpendicular to the xy plane and parallel to the y-axis are squares.



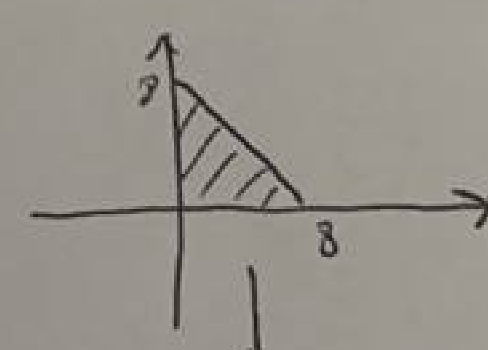
$$\text{Volume} = \int_0^1 (1-x^2)^2\,dx.$$

(74) Draw the region bounded by the curves. ~~Use~~ Find the volume when the region is rotated around the x-axis.

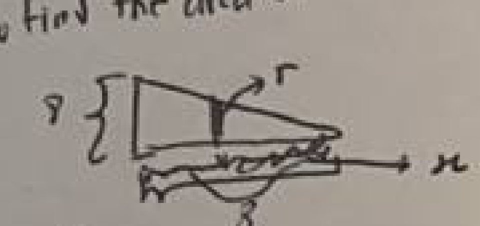
$x+y=8,\ x=0,\ y=0$

$\Downarrow$

$y=8-x.$



To find the area of this slice



$\therefore \text{Volume} = \int_0^8 \pi \cancel{r^2}\, dx$ $\quad (8-x)^2$

$= \cancel{\left[\frac{\pi x^3}{3}\right]_3^8}$

$= \cancel{\frac{512\pi}{3}} - \cancel{\frac{27\pi}{3}}$

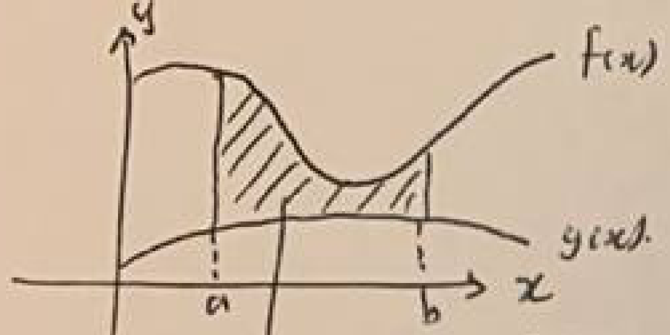
$\frac{r}{x} = \frac{8}{8}$ $\therefore r = x$ #.

So, given $0 \le x \le 8$, radius of circle formed at x is $8-x$.

$\Rightarrow$ Area of circle formed at x is $\cancel{\pi x^2}$ $\pi(x-8)^2$

Area ~~under~~ between Curves

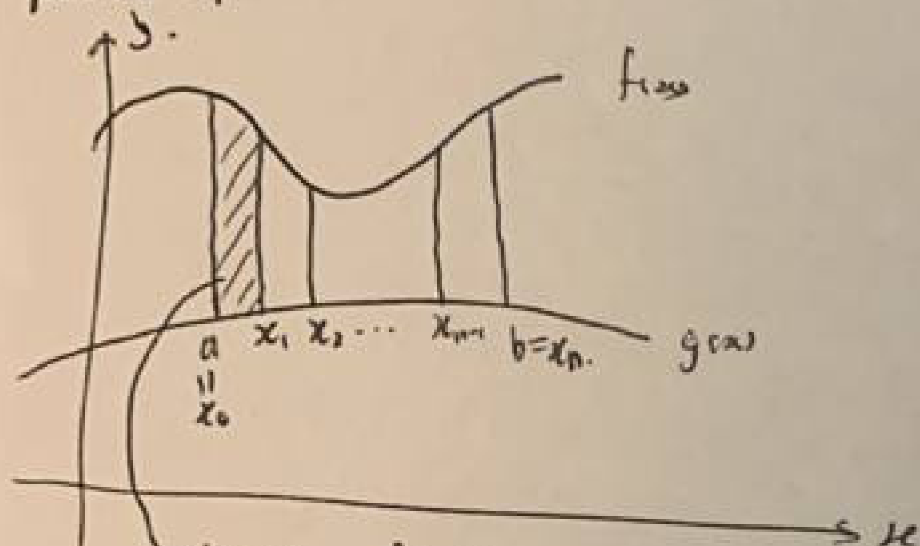
→ Area of a region between 2 curves.



↳ We want to find the area of this region.

→ As we did before, we are going to partition the interval on the ~~ax~~ x-axis and approximate the area between the graphs of the functions with rectangles.

→ So, for $i = 0, 1, 2, \dots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$.



↳ Height $= f(x_0) - g(x_0)$

base $= \Delta x$.

So, Area of this slice is $= [f(x_0) - g(x_0)]\Delta x$.

$\therefore$ Area between Curves $\simeq \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x$.

$\Rightarrow \text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \int_a^b f(x) - g(x)\, dx.$

⇒ This is a relatively simple case. When $a \le x \le b$, $f(x) > g(x)$.

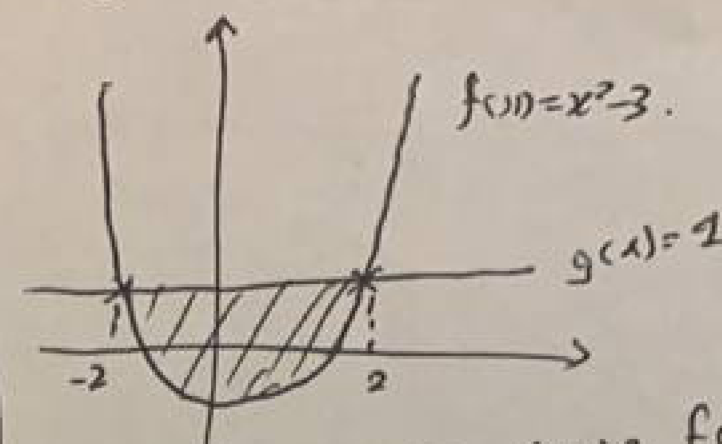
To generalise this,

$$\text{Area} = \int_a^b |f(x) - g(x)|\, dx.$$

→ To find Area of complex regions ← Break it down to simpler regions.

Qn 1: Find the area of the region between the 2 curves in the given figure by integrating with respect to the x-axis.

WICR4



→ Note that, $-2 \le x \le 2$, $g(x) \ge f(x)$.

By using the formula,

$$\int_{-2}^{2} g(x) - f(x)\, dx$$

$$= \int_{-2}^{2} 1 - (x^2 - 3)\, dx$$

$$= \int_{-2}^{2} 4 - x^2\, dx$$

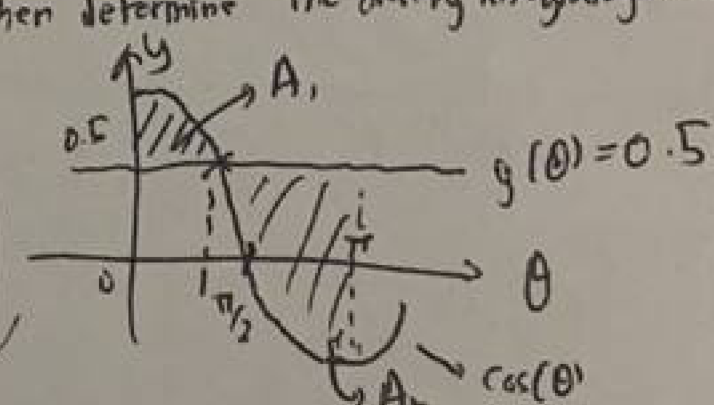
$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 8 - 8/3 + 8 - 8/3$$

$$= 16 - 16/3 = 32/3 \#.$$

Qn 4: Split the region between 2 curves into 2 smaller regions then determine the area by integrating over the x-axis.



Area of $A_1 = \int_0^1 \cos(\theta) - \frac{1}{2}\, d\theta$

$$= \left[\sin(\theta) - \frac{1}{2}\theta\right]_0^1$$

$$= \left(\sin(1) - \frac{1}{2}\right) - (\sin(0) - 0)$$

$$= \sin(1) - \sin(0) - 1/2$$

$$= \sin(1) - 1/2.$$

Area of $A_2 = \int_1^{\pi} \frac{1}{2} - \cos\theta\, d\theta = \left[\frac{1}{2}\theta - \sin\theta\right]_1^{\pi}$

$$= \left(\frac{\pi}{2} - \sin\pi\right) - \left(\frac{1}{2} - \sin 1\right)$$

$$= \frac{\pi}{2} - \sin(\pi) - 1/2 + \sin 1$$

Total Area $= A_1 + A_2$.

Midterm 1
Things to know

Limit Laws

① For any real number a and any constant c, (i) $\lim_{x\to a} x = a$

(ii) $\lim_{x\to a} c = c$.

② Sum laws for Limits: $\lim_{x\to a}(f(x)+g(x)) = \left(\lim_{x\to a} f(x)\right)+\left(\lim_{x\to a} g(x)\right)$

③ Difference law for Limits: $\lim_{x\to a}(f(x)-g(x)) = \left(\lim_{x\to a} f(x)\right)-\left(\lim_{x\to a} g(x)\right)$

④ Constant multiple Law for limits: $\lim_{x\to a} c\cdot f(x) = c\lim_{x\to a} f(x)$

⑤ Product Law for Limits: $\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$

⑥ Quotient Law: $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

⑦ Power Law for limits: $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$.

⑧ $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

Some techniques for finding Limits:

① Rationalising when we have $\frac{0}{0}$: For example $\lim_{x\to 4}\frac{\sqrt{x}-2\sqrt{x-3}}{12-3x}$ } Then, what we do is multiply by $\frac{\sqrt{x}+2\sqrt{x-3}}{\sqrt{x}+2\sqrt{x-3}}$

② Squeeze theorem: Let $f(x), g(x), h(x)$ be defined $\forall x\neq a$ over an open interval containing a. If

$$f(x)\le g(x)\le h(x)$$

for all $x\neq a$ in an open interval containing a and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, where L is a real number, then $\lim_{x\to a} g(x) = L$

③ Limits of Polynomials and rational functions. Let a be a real number. Then,

$$\lim_{x\to a} p(x) = p(a)$$

$$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a)\neq 0.$$

④ To show limits doesn't exist: $\lim_{x\to a} f(x)$ exist $\Leftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

Continuity.

Definition: A function $f(x)$ is continuous at a point a $\Leftrightarrow$ if and only if the following 3 conditions are satisfied:

(i) $f(a)$ is defined.
(ii) $\lim_{x\to a} f(x)$ exist.
(iii) $\lim_{x\to a} f(x) = f(a)$.

} Need to check all 3!

Types of Discontinuity:
→ Removable
→ infinite
→ Jump

Intermediate Value theorem: Let f be a continuous function over a closed, bounded interval $[a,b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ satisfying

$$f(c)=z.$$

Instantaneous Velocity: For a position function $s(t)$, the instantaneous velocity at a time $t=a$ is the value that the average velocity approaches on intervals of the form $[a,t]$ or $[t,a]$ as the values of t becomes close to a, provided a value exists.

Average velocity: Let $s(t)$ be the position of the object moving along a coordinate axis at time t. The average velocity over a time interval $[a,t]$ is

$$v_{ave} = \frac{s(t)-s(a)}{t-a}$$

Definition of Slope of tangent line: Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{tan} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \text{(OR) Alternatively} \quad m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

provided the limit exists.

Definition of Derivative: Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted $f'(a)$, is defined by

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \text{(OR) Alternatively} \quad f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Differentiation Rules

1. Let c be a constant. If $f(x)=c \Rightarrow f'(c)=0$. & $\frac{d}{dx}(c)=0$.
2. The power Rule: Let n be a positive integer. If $f(x)=x^n \Rightarrow f'(x)=nx^{n-1}$. Alternatively, we may express this rule as $\frac{d}{dx}x^n = nx^{n-1}$.
3. Sum Rule: Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then,
$$\frac{d}{dx}\left(f(x)+g(x)\right) = \left(\frac{d}{dx}f(x)\right)+\left(\frac{d}{dx}g(x)\right)$$
4. Difference rule: $\frac{d}{dx}\left(f(x)-g(x)\right) = \left(\frac{d}{dx}f(x)\right)-\left(\frac{d}{dx}g(x)\right)$
5. Constant Multiple Rule: $\frac{d}{dx}\left(k\,f(x)\right) = k\frac{d}{dx}f(x)$.
6. Product Rule: Let $f(x)$ and $g(x)$ be differentiable functions. Then, $\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}\left(f(x)\right)\cdot g(x) + \frac{d}{dx}g(x)\cdot f(x)$
7. Quotient Rule: Let $f(x)$ and $g(x)$ be differentiable functions. Then, $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}\left(f(x)\right)\cdot g(x) - \frac{d}{dx}\left(g(x)\right)\cdot f(x)}{(g(x))^2}.$

(1) The function $f(x)$ and its first and 2nd derivatives are

$f(x) = \sqrt{x^2-x-2}$

$$f'(x) = \frac{2x-1}{2\sqrt{x^2-x-2}}$$

$$f''(x) = -\frac{9}{4(x^2-x-2)^{3/2}}$$

1

M. Item 3

(A) Domain of $f(x)$:

x^2-x-2 can't be less than 0.

$\therefore x^2-x-2 \overset{=0}{} \Rightarrow x=-1$ or $x=2$.

Then, $x^2-x-2<0$ if $-1<x<2$.

So, the domain of $f(x)$: $(-\infty,-1] \cup [2,\infty)$.

(B) y-intercept and x-intercept.

When $y=0 \Rightarrow \sqrt{x^2-x-2}=0 \Rightarrow x^2-x-2=0 \rightarrow x=-1$ or $x=2$.

$(-1,0)$, $(2,0)$ are x intercepts.

When $x=0$, $f(x)=\sqrt{-2}$, so not possible.

No y-intercept.

(D) Intervals where $f(x)$ is increasing and decreasing. (Pg 392).

→ Use first derivative test.

(1) Solve for $f'(x)=0$.

$$\frac{2x-1}{2\sqrt{x^2-x-2}}=0 \Leftrightarrow 2x-1=0 \Leftrightarrow x=\tfrac{1}{2}.$$

But at $x=\frac{1}{2}$, $f(x)$ is not defined.

So, lets look at the 2 regions of the domain: $(-\infty,-1]$ and $[2,\infty)$.

take $x=-2 \Rightarrow$ $\dfrac{2(-2)-1}{2\sqrt{4+2-2}} = \dfrac{-5}{2\sqrt{4}} < 0.$

$\therefore f'(x)<0$. in the region $(-\infty,-1]$. } function is decreasing.

take $x=4 \Rightarrow$ $\dfrac{2(4)-1}{2\sqrt{16-4-2}} > 0.$

$\therefore f'(x)>0$ in the region $[2,\infty]$. } function is increasing.

(E) Intervals where $f(x)$ is concave up and concave down. (Pg 396).

Test for concavity: If $f''(x)>0$, $\forall x \in I \Rightarrow f$ is concave up.

If $f''(x)<0 \Rightarrow f$ is concave down.

(1) Find where $f''(x)=0$. $-\dfrac{9}{4(x^2-x-2)^{3/2}}=0 \Leftrightarrow -9=0$ (Not true).

$\therefore f''(x)=0$ no where in function f.

Def of inflection point: If f is ctn at a and change concavity at $a \Rightarrow (a, f(a))$ is an inflection point of f.

So, lets look at the 2 regions: $(-\infty, -1]$ and $[2,\infty)$

↳ take $x=-2$,

$$f''(-2) = \frac{-9}{4(4+2-2)^{3/2}} < 0$$

↳ $f''(3) = \dfrac{-9}{4(9-3-2)^{3/2}} < 0.$

(2) In each of the following parts, fill in "local max", "absolute max", "local min" or "absolute min".

→ This Qn uses the first derivative test. (Pg 392).

Suppose f is a ctn function over an interval I containing a critical pt c. If f is differentiable over I, except possibly at point c, $\Rightarrow f(c)$ satisfies one of the following descriptions:

(1) If f' changes from positive when $x<c$ to ~~positive~~ negative when $x>c \Rightarrow f(c)$ is a local max.

(2) If f' changes from negative when $x<c$ to positive when $x>c \Rightarrow f(c)$ is a local minimum of f.

(3) If $f'(x)$ has the same sign for $x<c$ and $x>c \Rightarrow f(c)$ is neither a local max nor a local min of f.

2nd derivative test.

Suppose $f'(c)=0$, f'' is ctn over an interval containing c.

(1) If $f''(c)>0 \Rightarrow f$ has a local min at c.

(2) $f''(c)<0 \Rightarrow f$ has a local max at c.

(3) If $f''(c)=0 \Rightarrow$ test is inconclusive

(A) local max. (if $h'(3)=0$, " $h''(3)=-2$)

(B) If $h'(-1)=0$, $h''(-2)=3$

$\Rightarrow$ local min.

(C) If $f'(0)=0$, $f'(x)<0$ on $(-\infty,0)$ and $f'(x)>0$ on $(0,\infty)$, $\Rightarrow f(x)$ has a global min

(D) $g'(-1)=0$ & $g''(2)=0$

$\Rightarrow$ test is inconclusive.

Midterm 3

(3) Find limit of $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$.

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \frac{\ln(\infty)}{\infty^2} = \frac{\infty}{\infty}.$$

So, we use L Hopital Rule. (pg 457).

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

~~$\lim_{x\to\infty} \frac{1}{x} = 0$~~

$$\frac{d}{dx} x^2 = 2x.$$

~~$\lim 2x = \infty$~~

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2}$$
$$= 0.$$

(B) $\lim_{x\to-\infty} \frac{1-e^{x}}{\sqrt{4e^{x}+1}} = \frac{1-e^{-\infty}}{\sqrt{4e^{-\infty}+1}}$

$$= \frac{1-\frac{1}{e^{\infty}}}{\sqrt{4\frac{1}{e^{\infty}}+1}} = 1$$

(C) $\lim_{x\to 1^+} (x-1)^{x-1} = (1-1)^{1-1} = 0^0.$

So, this is in indeterminate form.

Let $y = (x-1)^{x-1}$

$\Rightarrow$ we want to find. $\lim_{x\to1^+}(x-1)^{x-1}$.

So, $\lim_{x\to1^+} y = \lim_{x\to1^+} (x-1)^{x-1}$

$$\ln\left[\lim_{x\to1^+} y\right] = \ln\left[\lim_{x\to1^+}(x-1)^{x-1}\right]$$

$\ln\left[\lim_{x\to1^+} y\right] = 0$

So, $\lim_{x\to1^+} = e^0 = 1.$

Since $\ln$ is continuous,

RHS: $\lim_{x\to1^+} \ln(x-1)^{x-1} = \lim_{x\to1^+} (x-1)\ln(x-1)$

$$= \lim_{x\to1^+}(x-1)\left[\lim_{x\to1^+}\ln(x-1)\right]$$

$$= 0\cdot\ln\left[\lim_{x\to1^+} x-1\right].$$

$$= 0\cdot\infty \quad \text{Also indeterminant form.}$$

So, now we calculate $\lim_{x\to1^+} \frac{\ln(x-1)}{(1/x-1)} \xrightarrow[\text{Apply L'Hopital means}]{} \lim_{x\to1^+} \frac{(1/x-1)}{-1/(x-1)^2} = \lim_{x\to1^+} -(x-1) = 0.$

(4) Find the vertical and horizontal asymptotes of

$$f(x) = \frac{2x(x^2+1)^{1/2}}{x^2+8x+7} = \frac{(4x^4+4x^2)^{1/2}}{x^2+8x+7}.$$

$y=2$ is the ~~vertical~~ horizontal asymptote.

To find ~~horiz~~ vertical asymptote,

$f(x) = x^2+8x+7=0 \Leftrightarrow (x+7)(x+1)=0.$
$\Leftrightarrow x=-7$ or $x=-1$.

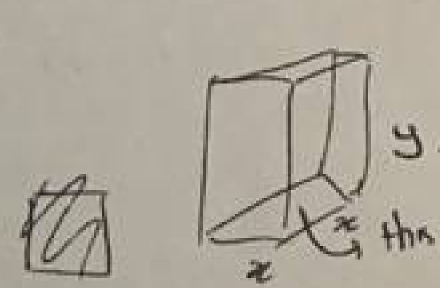
So, ~~x~~ denominator cannot be -7 or -1

Thus, vertical asymptotes occur when $x=-7$ and $x=-1$.

$$\lim_{x\to\infty} \frac{2x\sqrt{x^2+1}}{x^2+8x+7} = \frac{2x\sqrt{x^2}\sqrt{1+1/x^2}}{x^2(1+8/x+7/x^2)} = \frac{2x^2\sqrt{1+1/x^2}}{x^2(1+8/x+7/x^2)}$$

$$= \frac{2\sqrt{1+1/x^2}}{(1+8/x+7/x^2)} = \frac{2}{1} = 2.$$

(5) A rectangular open topped aquarium is to have square base and vol 8. The material for the base is \$2/m², the material for the sides \$1/m². What dimensions minimise the cost of the aquarium?



this is a square.

$x^2y = 8$

Cost of aquarium = Area of base $\times$ Cost of material for base + Total Area of height pieces $\times$ Cost of material

$$= x^2(2) + 4(xy)(1)$$
$$= 2x^2 + 4xy.$$

$f(x,y) = 2x^2 + 4xy.$

We want to minimize $f(x,y)$.

Step ①: We change it to ~~the~~ 1 variable using $x^2y=8$.

$$f(x) = 2x^2 + 4x\left(\frac{8}{x^2}\right) = 2x^2 + \frac{32}{x}.$$

$$f'(x) = 4x - 32x^{-2}$$

$$f'(x) = 0 \Leftrightarrow 4x = \frac{32}{x^2} \Leftrightarrow x^3 = 8$$
$$\Leftrightarrow x = 2.$$

~~Then~~ $f''(x) = 4 + 64x^{-3}$
$= 4 + 64(2)^{-3} > 0.$

So, using 2nd derivative test ~~H~~ at $x=2$,

we achieve the minimum cost.

File_002.png

6 Evaluate the following ~~indefinite~~ integrals.

Midterm 3

2

A $\int x + 2x^4 + 3x^{30}\,dx$

$= \int x\,dx + \int 2x^4\,dx + \int 3x^{30}\,dx$

$= \int x\,dx + 2\int x^4\,dx + 3\int x^{30}\,dx$

$= \frac{x^2}{2} + 2\left(\frac{x^5}{5}\right) + 3\left(\frac{x^{31}}{31}\right) + C.$

B $\int e^x + \frac{3+x^2}{1+x^2}\,dx = \int e^x\,dx + \int \frac{3+x^2}{1+x^2}\,dx$

$$\begin{array}{r} 1 \\ 1+x^2 \overline{)\, x^2+3} \\ -(x^2+1) \\ \hline 2 \end{array}$$

$= \int e^x\,dx + \int 1 + \frac{2}{1+x^2}\,dx$

$= \int e^x\,dx + \int 1\,dx + 2\int \frac{1}{1+x^2}\,dx$

$= e^x + x + 2\int (1+x^2)^{-1}\,dx. + C$

~~$= e^x + x + 2 \ln(1+x^2)$~~

$= e^x + x + 2\arctan(x) + C.$

C $\int \sqrt{x} + \frac{1}{x} + \sec^2(x)\,dx$

$= \int \sqrt{x}\,dx + \int \frac{1}{x}\,dx + \int \sec^2(x)\,dx.$

$= \frac{x^{3/2}}{3/2} + \ln|x| + \tan(x) + C.$

Finals

1. (A) $\lim_{x\to 0} \frac{x+4}{x+1} = \frac{\lim_{x\to 0} x+4}{\lim_{x\to 0} x+1}$

$= \frac{4}{1} = 4.$

(B) $\lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta^2} = \frac{\sin(0)}{0} = \frac{0}{0}$ } indeterminate form.

So, using L hopital Rule,

$\lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta^2} = \lim_{\theta\to 0} \frac{\cos(\theta^2)(2\theta)}{2\theta} = \lim_{\theta\to 0} \cos(\theta^2)$

$= \cos(0) = 1.$

(C) $\int \left(x^{1/3} + \sin(x) + \frac{1}{x}\right) dx = \int x^{1/3}\, dx + \int \sin(x)\, dx + \int \frac{1}{x}\, dx$

$= \frac{x^{1/3+1}}{(1/3+1)} + (-\cos(x)) + \ln|x| + C.$

$= \frac{3}{4} x^{4/3} - \cos(x) + \ln|x| + C.$

(D) $\frac{d}{dx} \int_3^{x^2} \sin(t+2)\, dt.$

Let $u(x) = \cancel{x}\, x^2$, and $F(x) = \int_3^{u(x)} \sin(t+2)\, dt.$ $F'(x) = \frac{dF(x)}{dx} = \frac{dF}{du} \cdot \frac{du}{dx}$

$= \sin(u(x)+2) \cdot (2x)$ } using Fundamental theorem of Calculus.

$= \sin(x^2+2)(2x).$

(E) $\frac{d}{dx}\left(\frac{\ln(x)}{e^x + \sec(x)}\right) = \frac{d}{dx}\left(\ln(x)\,[e^x + \sec(x)]^{-1}\right)$

$= \frac{1}{x}[e^x + \sec(x)]^{-1} + \ln(x)(-1)[e^x + \sec(x)]^{-2}(e^x + \sec(x)\tan(x))$

(F) $\frac{d}{dx}\left(\cos(x^2) \cdot \arcsin(x)\right) = \left[\frac{d}{dx}\cos(x^2)\right]\arcsin(x) + \cos(x^2)\frac{d}{dx}\arcsin(x)$

$= -\sin(x^2)(2x)\arcsin(x) + \cos(x^2)\left(\frac{1}{\sqrt{1-x^2}}\right)$

(8) Find the absolute ~~rel~~ maximum of $w(x) = x + \sin(x)$ in the interval $[0, 2\pi]$.

→ Note that f is defined over the closed interval $[a,b]$. So,

Step ①: Evaluate $w(x)$ at the endpoints.

Step ②: Find all critical pts of ~~or~~ f over the interval $f(a,b)$, evaluate f at those pts.

Step ③: Pick the largest value.

To find critical values:

② $\frac{dw}{dx} = 1 + \cos(x)$.

Solving for $1 + \cos(x) = 0$

$\cos(x) = -1$

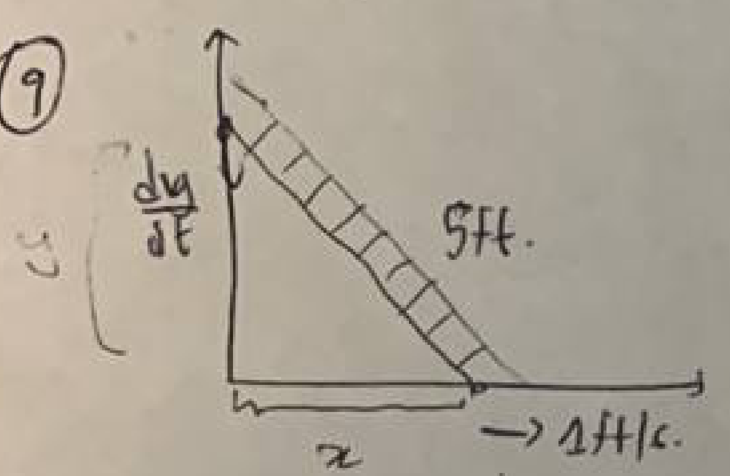
$x = \cos^{-1}(-1)$

$x = \pi$.

~~w~~ x	$w(x)$
0	0
2π	2π
π	π

∴ Max happens at $x = 2\pi$

(9)



$\frac{dx}{dt} = 1$.

$x^2 + y^2 = 25$. ∴ $y = \sqrt{25 - x^2}$.

$\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-1/2}(-2x)$

$= -x(25 - x^2)^{-1/2}$.

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

∴ $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= -x(25 - x^2)^{-1/2}$.

when $x = 3$ft,

$\frac{dy}{dt} = -3(16)^{-1/2} = -3/4$ #

(6) Growth of a dog in pounds from week 2 to 6.

(7) $u(x) = \sqrt{x}, \quad x = 9.$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$\cancel{u(9) = \sqrt{9} = 3}.$

$L(x) = u(9) + u'(9)(x-9)$

$\cancel{y - 3 = \frac{1}{2\sqrt{x}}(x-9)}$

$u'(9) = \frac{1}{2\sqrt{9}} = 1/6.$

$\cancel{y = \frac{x}{2\sqrt{x}} - \frac{9}{2\sqrt{x}} + 3}$

$\therefore L(x) = 3 + \frac{1}{6}(x-9)$

$\cancel{y = \frac{x-9}{2\sqrt{x}} + 3}.$

So, $\sqrt{9.2} = u(9.2)$

$\approx L(9.2)$

$= 3 + 1/6\,(9.2 - 9)$

$= 3 + \frac{0.2}{6}.$

~~So when $x = 9.2$,~~

(5) (A) $\int \frac{\ln(x)^4}{x} dx$.

Let $u = \ln(x)$.

$\frac{du}{dx} = \frac{1}{x}$

$\Rightarrow du = \frac{1}{x} dx$

$\Rightarrow x \, du = dx$.

So, $\int \frac{\ln(x)^4}{x} dx = \int u^4 \, du$

$= \frac{u^5}{5} + C$

$= \frac{\ln(x)^5}{5} + C$.

(B) $\int_0^1 \frac{1}{\sqrt{25-4x^2}} dx = \int_0^1 \frac{1}{\sqrt{5^2-(2x)^2}} dx = \sin^{-1}\left(\frac{2x}{5}\right) + C$.

(2) Using the limit definition of the derivative, find $f'(2)$ if $f(x) = \frac{1}{\sqrt{x}}$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right)$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right)$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x(x+h)}}\right)\left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right) \frac{x-x-h}{\sqrt{x(x+h)}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \lim_{h\to 0} \cancel{\frac{-h}{\sqrt{x(x+h)}}\left(\frac{1}{h}\right)} \quad \frac{-h}{(\sqrt{x}+\sqrt{x+h})(\sqrt{x(x+h)})}\left(\frac{1}{h}\right)$$

$$\cancel{= \lim_{h\to 0} \frac{-1}{\sqrt{x(x+h)}} = \frac{-1}{\sqrt{x^2}}}$$

$$= \lim_{h\to 0} -\frac{1}{(\sqrt{x}+\sqrt{x+h})(\sqrt{x(x+h)})}$$

$$= -\frac{1}{2\sqrt{x}\sqrt{x^2}} = \frac{-1}{2\sqrt{x^3}} \ \#$$

$$\frac{d}{dx}\frac{1}{\sqrt{x}} = \frac{d}{dx} x^{-1/2}$$
$$= -\frac{1}{2}x^{-3/2}$$

So, when $x=2$, $f'(2) = -\frac{1}{2\sqrt{2^3}} = -\frac{1}{2\sqrt{8}}$ #

(3) Implicit Differentiation.

(A) $x^2 - 3xy + y^2 = 5$.

Differentiate with respect to x,

$$2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(-3x+2y) = 3y - 2x.$$

So, $\frac{dy}{dx} = \frac{3y-2x}{-3x+2y}$.

(B) $y = x^{x^2}$

$\ln(y) = \ln(x^{x^2})$

$\Rightarrow \ln(y) = x^2\ln(x)$

Differentiating wrt x,

$$\frac{1}{y}\frac{dy}{dx} = 2x\ln(x) + x^2\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = y\,(2x\ln(x) + x)$$

$$\Rightarrow \frac{dy}{dx} = x^{x^2}(2x\ln(x) + x).$$

(4) If $f''(x) > 0, \forall x \in I \Rightarrow f$ is concave up over I.

If $f''(x) < 0\ \forall x \in I \Rightarrow f$ is concave down over I.

(A) Solve $w''(x) = \frac{x+1}{\sqrt{x^2+2}}$.

$$\frac{x+1}{\sqrt{x^2+1}} > 0.$$

Solve for $\frac{x+1}{\sqrt{x^2+1}} = 0$

$\Leftrightarrow x = -1.$

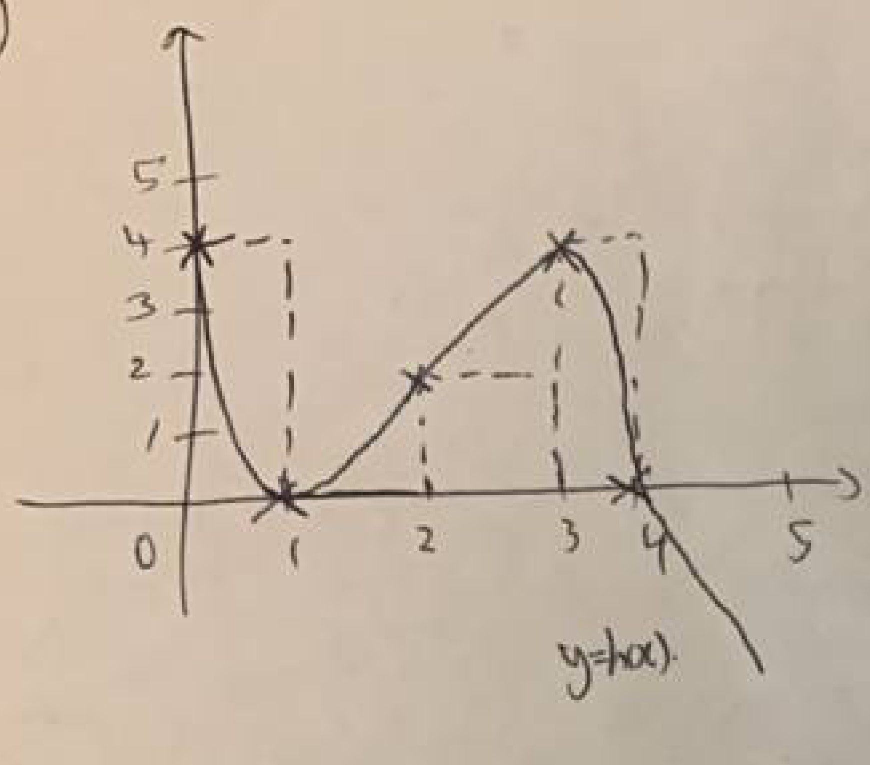
-2. -1 2.

$w''(-2) < 0$ Concave down

$f''(-1) = 0$ inflection point

$w''(2) > 0$ concave up.

(10)



Estimate $\int_0^4 h(x)dx$ by using riemann sum with subintervals $n=4$, taking the sample points to be the left endpoint (the Left Hand rule L_4).

$x_0=0,\ x_1=1,\ x_2=2,\ x_3=3.$

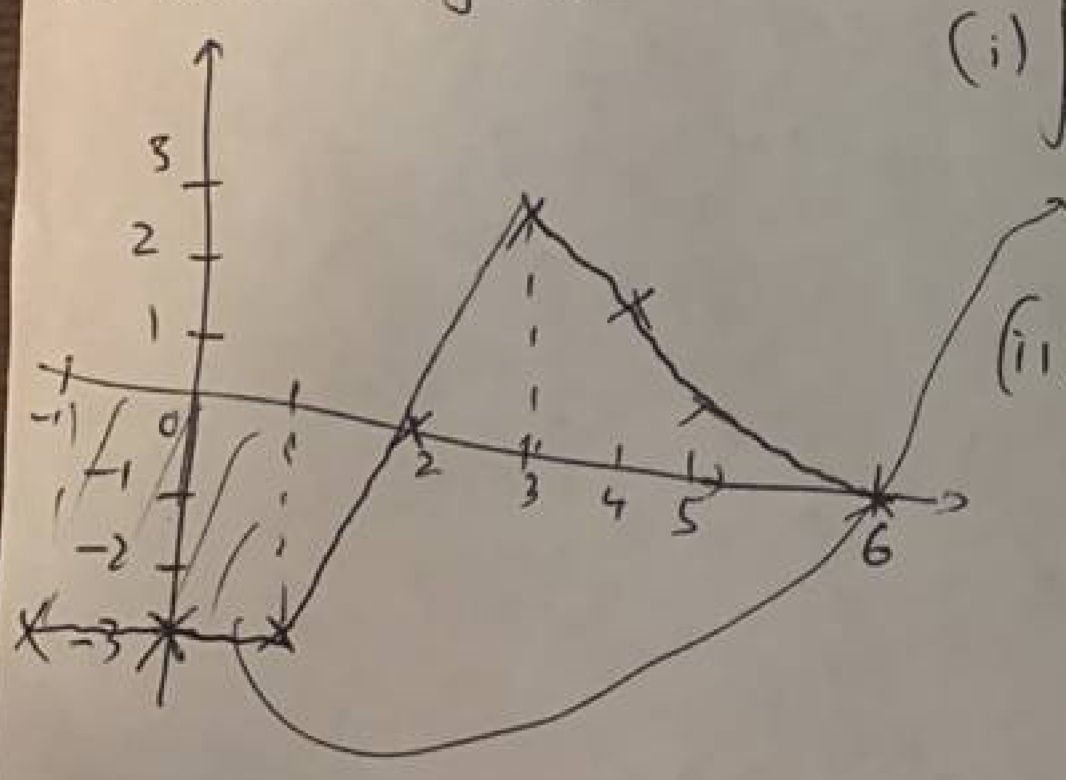
$f(x_0)=4,\ f(x_1)=0,\ f(x_2)=2,\ f(x_3)=4.$

$\Delta x=1$

$$\therefore \int_0^4 h(x)\,dx \approx 4(1)+0(1)+2(1)+4(1)$$

$$=10.$$

(11) Evaluate the following integrals.



(i) $\int_1^{-1} g(x)\,dx = -\int_{-1}^{1} g(x)\,dx$

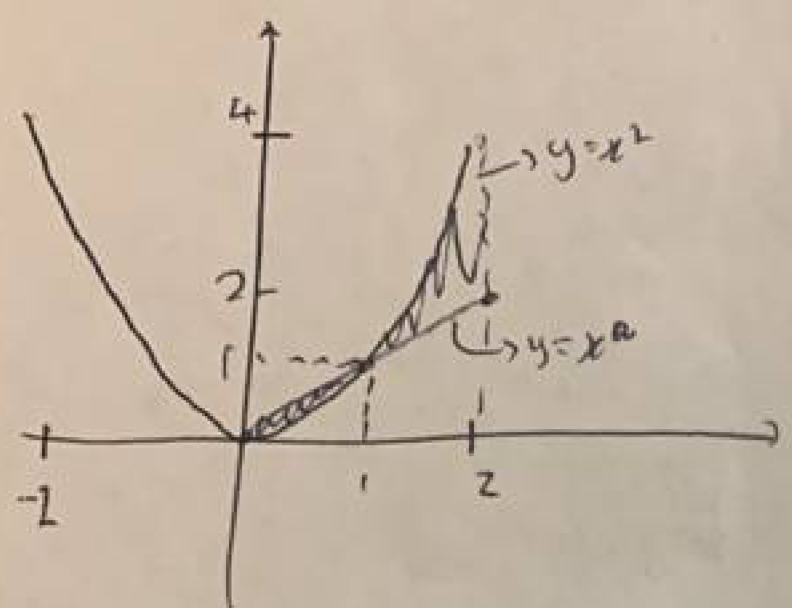
$= 2\times3 = 6$

(ii) $\int_1^5 g(x)\,dx = \int_1^2 g(x)\,dx + \int_2^3 g(x)\,dx + \int_3^5 g(x)\,dx$

$= -\frac{1}{2}(1)(3) + \frac{1}{2}(1)(3) + \frac{1}{2}(3+1)(2)$

$= 4$ #.

(12) Find the Area bounded between $y=x$ and $y=x^2$ between 0 and 2.

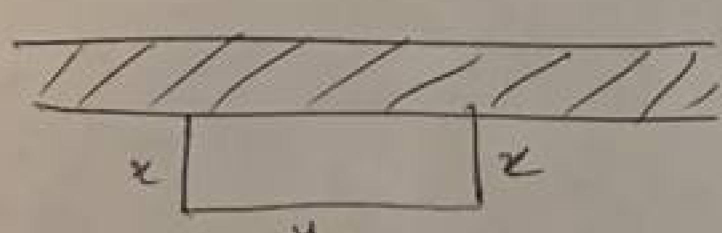


Area shaded $= \int_0^1 x-x^2\,dx + \int_1^2 x^2-x\,dx$

$$= \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2$$

$$= \left[\frac{1}{2}-\frac{1}{3}\right] + \left[\frac{8}{3}-2\right] \#$$

(13) A Farmer has 15ft of fencing and wants to fence off a rectangular area that borders a straight river. The Farmer needs no fencing along the river. What dimensions will maximise the fenced in area?



$2x+y=15$.

Maximise Area.

Area $= xy$.

$= x(15-2x)$.

$\therefore \frac{dA}{dx} = (15-2x) + x(-2)$

$= 15-4x$.

$\frac{dA}{dx} = 0 \iff x = 15/4$.

Use ~~the~~ 1st derivative test to confirm if this is the max.

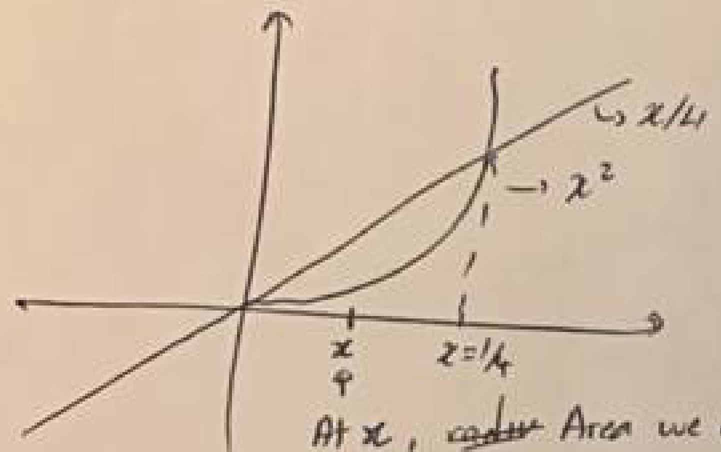
$x = 15/4 \Rightarrow \frac{dA}{dx} = 0$

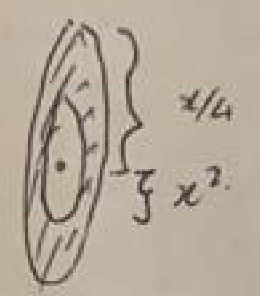
$x=0 \Rightarrow \frac{dA}{dx} > 0$

$x = 5 \Rightarrow \frac{dA}{dx} < 0$

$\therefore x = 15/4$ is the maximum point.

(14) Find the volume of the solid obtained by rotating the region bounded by $y = x/4$ & $y = x^2$ around the x-axis





At x, ~~radius~~ Area we need

So, Area at $x = \pi\left(\frac{x}{4}\right)^2 - \pi(x^2)^2$

$$\therefore \int_0^{1/4} \pi\left(\frac{x}{4}\right)^2 - \pi(x^2)^2 \, dx = \frac{\pi}{7680} \#$$